

An example of a non-sofic hyperbolic group

ABSTRACT. We construct a torsion-free, two-dimensional, word-hyperbolic group which is not sofic. Our construction builds on OpenAI's construction of a non-residually finite hyperbolic group [14].

1. Introduction

For a positive integer n , let $\text{Sym}(n)$ denote the symmetric group on $\{1, \dots, n\}$. Its normalized Hamming metric and the fixed-point fraction of a permutation are given by

$$d_H(\sigma, \rho) = \frac{|\{a : \sigma(a) \neq \rho(a)\}|}{n}, \quad \text{fix}(\sigma) = \frac{|\{a : \sigma(a) = a\}|}{n}.$$

We write id for the identity permutation. A group G is sofic if, for every finite subset $F \subset G$ containing 1 and every $\eta > 0$, there are a positive integer n and a map $\phi : F \rightarrow \text{Sym}(n)$, sending 1 to the identity permutation, such that

$$d_H(\phi(a)\phi(b), \phi(ab)) < \eta \quad (a, b, ab \in F), \quad \text{fix}(\phi(a)) < \eta \quad (a \in F \setminus \{1\}).$$

Thus multiplication is preserved on most points of the finite set, while each specified nonidentity element fixes only a small proportion of the points. The notion was introduced by Gromov [8]; see also Weiss [19]. All amenable groups and all residually finite groups are sofic; see [16]. The question whether all groups are sofic was open for a long time.

The first proof of the existence of a non-sofic group was given by OpenAI [15]. Kun and Thom [11] constructed non-sofic generalized wreath products and doubles of residually finite groups. Chapman and Peled [4] gave a conditional construction of a non-sofic hyperbolic group using random simplicial complexes. The aim of this paper is to construct a torsion-free word-hyperbolic group which is not sofic. We build on OpenAI's construction of a non-residually finite torsion-free hyperbolic group [14]. Recall that a finitely generated group is word-hyperbolic if geodesic triangles in a Cayley graph for a finite generating set are uniformly thin: each side lies within a fixed distance of the other two sides. A group is torsion-free if the identity is its only element of finite order.

We obtain the group as the fundamental group of a finite complex. By a triangle complex we mean a two-dimensional cell complex obtained from a graph by attaching triangles, each side being mapped homeomorphically onto an edge. Several edges may have the same endpoints. In our construction, each closed face is embedded. We assign a length to every edge and give the faces the metrics of triangles in the hyperbolic plane of curvature -1 , with the prescribed side lengths. The complex carries the induced path metric.

A geodesic metric space is $\text{CAT}(-1)$ if distances between points on the sides of a geodesic triangle are at most the distances between the corresponding points of a triangle in the hyperbolic plane with the same side lengths. Corresponding points have the same distances from the endpoints of the corresponding sides. The metric is locally $\text{CAT}(-1)$ if each point has a neighbourhood which, with its intrinsic path metric, is $\text{CAT}(-1)$. We refer to [3] for these notions and their basic properties. Our main result is the following.

THEOREM A. *There exists a finite connected two-dimensional triangle complex K with a locally $\text{CAT}(-1)$ metric such that $G = \pi_1(K)$ is torsion-free, word-hyperbolic, and non-sofic.*

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Put $q = 9 \cdot 10^{22} + 1$. The presentation of G in Section 3 has

$$\begin{aligned} 10^{12}(q-1)q^4 + 6 \cdot 10^{17}q^5 - 2 &< 4 \cdot 10^{132} && \text{generators,} \\ 10^{12}(q-1)q^5 - 10^7(q-1)^5 &< 6 \cdot 10^{149} && \text{relations.} \end{aligned}$$

The group G in Theorem A admits no proper cocompact action by cubical automorphisms on a CAT(0) cube complex. Indeed, by Agol's theorem [1, Theorem 1.1] and the work of Haglund and Wise [9, Theorem 4.4], a word-hyperbolic group admitting such an action is residually finite and therefore sofic.

COROLLARY B. *There exists a nontrivial torsion-free word-hyperbolic group which has no nontrivial sofic quotient.*

PROOF. If every nontrivial torsion-free word-hyperbolic group had a nontrivial sofic quotient, the theorem of Kapovich and Wise [10, Theorem 3.4], applied to the class of sofic groups, would imply that every torsion-free word-hyperbolic group is residually sofic and hence sofic by [7]. This contradicts Theorem A. $\square$

The construction starts with a family of affine lines in $\mathbb{F}_q^5$, where $\mathbb{F}_q$ is the field with q elements and q is prime. Its incidence graph has one vertex for each point and each line, and an edge whenever the point lies on the line. This is the graph $D(5, q)$ of Lazebnik and Ustimenko [12]. Its girth, the length of its shortest simple cycle, is at least ten [13, Proposition 2.1(iii)]. In Section 2, we select a family of lines with distinct distinguished points, called marks, and delete the incidence with its own mark for each line in a specified subfamily.

The retained incidences determine a triangle complex with three vertices. Each retained line gives an array of edges between two of these vertices. Choosing two rows and two columns gives a closed four-edge path, traversed with alternating orientations.

In Section 3, we combine polynomial interpolation with the rank estimate of Alon–Hall–Knauer–Pinchasi–Yuster [2, Theorem 6]. We prove that every assignment of permutations to the oriented edges has either a triangular boundary word moving a fixed positive fraction of points or one of the four-edge words fixing such a fraction. The bound is independent of the degree of the permutations.

In Section 4, we use the girth bound to construct a locally CAT(−1) metric and prove that the four-edge paths are non-nullhomotopic. Soficity would allow both fractions in the preceding paragraph to be arbitrarily small. Section 5 derives the contradiction.

Acknowledgments. This paper is based on an AI-generated construction obtained by Harjas Sandhu on 7 October 2026. The second author worked through the proof to make it readable and to reduce unnecessary overhead. The aim of this note is to give a reasonably self-contained presentation of the result. An independent AI-assisted proof of the existence of a non-sofic hyperbolic group was given by Seth Douglas and Nidhal Mghirbi [5].

The construction is inspired by OpenAI's solution of the existence problem for a non-residually finite torsion-free hyperbolic group [14]. From that construction we use the marked lines, triangular relations associated to Cartesian products, interpolation identity, and comparison of real and finite-field ranks. We prove these ingredients and combine them with the published rank theorem and the incidence relations of $D(5, q)$ to obtain the inequality for approximate permutation assignments. We do not claim novelty for the rank theorem or the interpolation identity. I thank Ian Agol for pointing out the relevance of Dvir's work [6] to the crucial interpolation argument, see Lemma 2.1.

2. Marked lines from $D(5, q)$

We fix

$$r = 10^6, \quad h = 3 \cdot 10^{11}, \quad q = h^2 + 1 = 9 \cdot 10^{22} + 1, \quad s = \frac{q-1}{10}.$$

The integer q is prime.¹ We consider two copies of $\mathbb{F}_q^5$, points and labels. We write points as (t, y, z, w, v) and labels as $[a, b, c, d, e]$. The incidence relation between points and labels is defined by

$$b - y = at, \quad c - z = bt, \quad d - w = ay, \quad e - v = az. \quad (1)$$

¹We have $q-1 = 2^{22}3^25^{22}$. Direct modular exponentiation gives $11^{q-1} \equiv 1 \pmod{q}$ and $\gcd(11^{(q-1)/p} - 1, q) = 1$ for $p \in \{2, 3, 5\}$. These identities force 11 to have order $q-1$ modulo every prime divisor ℓ of q . Hence $q-1$ divides $\ell-1$, so $\ell = q$.

These equations define a bipartite graph $D(5, q)$ as in [13, Section 2]. By [13, Proposition 2.1], this graph is q -regular, has q^5 vertices in each part, and has girth at least ten. These properties also follow from [12, Proposition 2.1 and Theorem 3.3].

For a fixed label, the incident points form the full affine line

$$\ell_{[a,b,c,d,e]} = \{(t, b - at, c - bt, d - ab + a^2t, e - ac + abt) : t \in \mathbb{F}_q\}. \quad (2)$$

Different labels determine different lines. Indeed, the normalized direction $(1, -a, -b, a^2, ab)$ determines a, b , and the point with first coordinate zero then determines c, d, e .

Consider the set of incidence edges satisfying $a = t$. Given a point (t, y, z, w, v) , the condition $a = t$ and the incidence equations uniquely determine a label by

$$a = t, \quad b = y + t^2, \quad c = z + bt, \quad d = w + ty, \quad e = v + tz.$$

Thus each point is contained in exactly one of these incidence edges. Conversely, for a fixed label $[a, b, c, d, e]$, the parametrization (2) gives exactly one incident point with first coordinate $t = a$. Thus each label is also contained in exactly one of these incidence edges. These incidence edges therefore form a perfect matching. Since each matched pair satisfies (1), the point matched to a label lies on the corresponding affine line. We henceforth index each line by its matched point, writing ℓ_i for the line marked at $i \in \mathbb{F}_q^5$.

Put $S = \{0, \dots, s-1\} \subset \mathbb{F}_q$, using the standard integer representatives, and set

$$I = S^5, \quad J = (\mathbb{F}_q \setminus \{-1\}) \times \mathbb{F}_q^4.$$

We retain all lines ℓ_i with $i \in J$ and distinguish those with $i \in I$. For each distinguished line, we delete only its incidence with its own mark. All incidences of the other retained lines remain. Thus, for $x \in \mathbb{F}_q^5$, the remaining incidences at x are indexed by

$$J(x) = \{i \in I : x \in \ell_i \setminus \{i\}\} \cup \{i \in J \setminus I : x \in \ell_i\}. \quad (3)$$

The bipartite graph Γ on $\mathbb{F}_q^5 \sqcup J$, whose two parts represent points and retained lines, has an edge from x to i precisely when $i \in J(x)$. It is a subgraph of $D(5, q)$ and therefore has girth at least ten.

For a fixed point and a prescribed first label coordinate a , (1) uniquely determines b, c, d, e . The first coordinate of the matched point is a , so the retained lines are precisely those with $a \neq -1$. Thus $q-1$ retained lines pass through each point before deletion. A point loses one incidence if it belongs to I , and none otherwise. Consequently,

$$|J(x)| = \begin{cases} q-2 = (h-1)(h+1), & x \in I, \\ q-1 = h \cdot h, & x \notin I. \end{cases}$$

For each point x , put $(s_x, t_x) = (h-1, h+1)$ if $x \in I$ and $(s_x, t_x) = (h, h)$ otherwise. All these factors are at least r . Choose a bijection

$$J(x) \longrightarrow (\mathbb{Z}/s_x\mathbb{Z}) \times (\mathbb{Z}/t_x\mathbb{Z}), \quad i \longmapsto (\alpha_x(i), \beta_x(i)). \quad (4)$$

The sets $J(x)$ may overlap, but their bijections are chosen separately for each point.

The following interpolation argument is adapted from [14, Section 2.5, equation (2.13)] to the grid S^5 . It uses low-degree polynomials on affine lines, as in Dvir's proof of the finite-field Kakeya conjecture [6], applied here to the entries of rank-one matrices.

LEMMA 2.1. *There are matrices $H_x \in M_I(\mathbb{F}_q)$, indexed by $x \in \mathbb{F}_q^5$, such that*

$$\text{rank } H_x \leq 1, \quad \sum_{x: i \in J(x)} H_x = \begin{cases} -E_{ii}, & i \in I, \\ 0, & i \in J \setminus I. \end{cases} \quad (5)$$

PROOF. For $c \in S$, let p_c be the Lagrange basis polynomial associated with the node c in S , given by

$$p_c(T) = \prod_{d \in S \setminus \{c\}} \frac{T-d}{c-d}.$$

It is the unique polynomial of degree at most $s-1$ satisfying

$$p_c(d) = \begin{cases} 1, & d = c, \\ 0, & d \in S \setminus \{c\}. \end{cases}$$

For $i = (c_{i1}, \dots, c_{i5}) \in I$ and $X = (X_1, \dots, X_5)$, set

$$f_i(X) = \prod_{j=1}^5 p_{c_{ij}}(X_j), \quad f = (f_i)_{i \in I}, \quad H_x = f(x)f(x)^\top.$$

We have $\text{rank } H_x \leq 1$. For $i, i' \in I$, $f_i(i')$ equals 1 if $i = i'$ and 0 otherwise, so $H_i = E_{ii}$ for $i \in I$. Each f_i has total degree at most $5(s-1)$. It follows that every entry of $ff^\top$, restricted to an affine line, is a polynomial of degree at most $10(s-1) = q-11 < q-1$.

It is elementary to see that the sum of a polynomial of degree less than $q-1$ over $\mathbb{F}_q$ is zero. Applying this observation entrywise gives

$$\sum_{x \in \ell_i} H_x = 0 \quad (i \in J).$$

For $i \in I$, deleting the term at its mark gives $-H_i = -E_{ii}$. For $i \in J \setminus I$, the full line is retained, so its sum remains zero. This proves (5). $\square$

3. A finite permutation obstruction

Our next aim is to define a connected triangle complex whose fundamental group will have all the required properties. We use the construction in Section 2 and write $[r] = \{0, \dots, r-1\}$. We follow the construction in [14, Section 2.3]. Consider three vertices O, V, W and define oriented edges

$$A_{i,u,v} : V \longrightarrow W, \quad L_{x,u,j} : O \longrightarrow V, \quad R_{x,v,k} : O \longrightarrow W.$$

Here $i \in J$, $u, v \in [r]$, $x \in \mathbb{F}_q^5$, $j \in \mathbb{Z}/s_x\mathbb{Z}$, and $k \in \mathbb{Z}/t_x\mathbb{Z}$. For each point x , $i \in J(x)$, and $u, v \in [r]$, attach a triangle with boundary, read in path order,

$$L_{x,u,\alpha_x(i)+v} A_{i,u,v} R_{x,v,\beta_x(i)+u}^{-1}. \quad (6)$$

Denote the resulting finite triangle cell complex by K . There are L - and R -edges, and hence K is connected.

Let $G = \pi_1(K, O)$. Choose a point $x_0 \in \mathbb{F}_q^5$. The edges $c_V := L_{x_0,0,0}$ and $c_W := R_{x_0,0,0}$ form a spanning tree of the graph of vertices and edges of K . Let c_O be the constant path at O . For an oriented edge $e : A \rightarrow B$, define

$$g_e = [c_A e c_B^{-1}] \in G. \quad (7)$$

Denote the elements associated with $A_{i,u,v}$, $L_{x,u,j}$, and $R_{x,v,k}$ by $a(i, u, v)$, $l(x, u, j)$, and $r(x, v, k)$, respectively. We multiply loops in path order. Collapsing the spanning tree gives a finite presentation of G with generators $a(i, u, v), l(x, u, j), r(x, v, k)$, with the index ranges specified above, and relations

$$l(x, u, \alpha_x(i) + v) a(i, u, v) r(x, v, \beta_x(i) + u)^{-1} = 1 \quad (8)$$

for every point x , $i \in J(x)$, and $u, v \in [r]$, together with

$$l(x_0, 0, 0) = 1, \quad r(x_0, 0, 0) = 1.$$

Our next aim is to understand how this presentation interacts with the assignment of permutations to the generators. We identify a permutation $\sigma \in \text{Sym}(n)$ with the matrix P_σ defined by $P_\sigma e_a = e_{\sigma(a)}$, where e_a is the a -th standard coordinate vector. Thus inverse matrices are transposes, and $\text{tr}(P_\sigma) = n \text{fix}(\sigma)$. Assign arbitrary permutations of degree n to the oriented edges of K . Denote the permutations assigned to $A_{i,u,v}$, $L_{x,u,j}$, and $R_{x,v,k}$ by $A(i, u, v)$, $L(x, u, j)$, and $R(x, v, k)$, respectively. Reversing an edge replaces its permutation by its inverse. For a triangular face indexed by $\tau = (x, i, u, v)$, define

$$\delta_{x,i,u,v} = d_H(L(x, u, \alpha_x(i) + v) A(i, u, v) R(x, v, \beta_x(i) + u)^{-1}, \text{id}). \quad (9)$$

This is exactly the defect of the corresponding relation being satisfied in the permutation group. For $i \in I$, $u \neq u'$ and $v \neq v'$, define the rectangular product

$$Q_i(u, u', v, v') = A(i, u, v) A(i, u', v)^{-1} A(i, u', v') A(i, u, v')^{-1}. \quad (10)$$

Let λ_i be the average of $\text{fix}(Q_i(u, u', v, v'))$ over all $r^2(r-1)^2$ ordered tuples (u, u', v, v') with $u \neq u'$ and $v \neq v'$.

The key technical result is the following theorem. It gives a lower bound on the sum of defects of the defining relations together with a measure of non-triviality of the rectangular words $w_{i;u,u',v,v'} :=$

$a(i, u, v)a(i, u', v)^{-1}a(i, u', v')a(i, u, v')^{-1}$. We will later show that all these rectangular words are non-trivial in G .

THEOREM 3.1. *Every such assignment, in every degree $n \geq 1$, satisfies*

$$\frac{q^5}{5} \leq \sum_{\tau} \delta_{\tau} + \frac{r^2}{8} \sum_{i \in I} \lambda_i. \quad (11)$$

The proof of this theorem occupies the remainder of this section and needs various lemmas. The following result appears in [14, Lemma 2.4]. It is also a consequence of Gabber's bound on torsion in cokernels; see [18, Lemma 1] and [17, Lemma 3.1]. We include a direct proof.

LEMMA 3.2. *If an integer matrix M has row norms at most $\sqrt{r}$ and $q > r$ is prime, then*

$$\text{rank}_{\mathbb{F}_q} M \geq \frac{1}{2} \text{rank}_{\mathbb{R}} M.$$

PROOF. Set $d = \text{rank}_{\mathbb{R}} M$ and choose a nonsingular $d \times d$ minor B if $d > 0$. By Hadamard's inequality, we have $1 \leq |\det B| \leq r^{d/2}$. Let $d' = \text{rank}_{\mathbb{F}_q} B$ and choose d' rows forming a basis modulo q . Subtract integer lifts of their linear combinations from the remaining rows. These row operations preserve the determinant and make $d - d'$ rows divisible by q . Hence $q^{d-d'}$ divides $\det B$, so

$$q^{d-d'} \leq r^{d/2}.$$

Since $q > r$, this implies $d' \geq d/2$. If $d = 0$, there is nothing to prove. $\square$

LEMMA 3.3. *For $i \in J$, let $Y_i = (A(i, u, v))_{u, v \in [r]}$, an $rn \times rn$ integer block matrix. For every $i \in I$,*

$$\text{rank}_{\mathbb{F}_q} Y_i \geq \frac{r^2 n}{2(2r - 1 + (r - 1)^2 \lambda_i)} \geq n \left(\frac{r}{4} - \frac{r^2 \lambda_i}{8} \right). \quad (12)$$

PROOF. We apply [2, Theorem 6]. For a nonzero 0-1 matrix A , this theorem gives the following estimate in terms of the bipartite graph whose vertices are the rows and columns and whose edges correspond to entries equal to 1:

$$\text{rank}_{\mathbb{R}} A \geq \frac{e^2}{\sum_x d_x^2 - e + 4c_4}. \quad (13)$$

Here e is the number of edges, d_x the vertex degrees, and c_4 the number of unoriented four-cycles.

For $A = Y_i$, each of the $2rn$ vertices has degree r and $e = r^2 n$. Every four-cycle uses two different row blocks and two different column blocks: two rows of one permutation block cannot share a column, and the analogous assertion holds for columns. For each ordered choice of these four blocks, the cycles are counted by the fixed points of (10). Each unoriented cycle is counted four times, giving $4c_4 = nr^2(r - 1)^2 \lambda_i$. By (13), we get

$$\text{rank}_{\mathbb{R}} Y_i \geq \frac{r^2 n}{2r - 1 + (r - 1)^2 \lambda_i}.$$

Each row of Y_i has exactly r entries equal to 1, so Lemma 3.2 proves the first bound in (12).

For the second inequality, use $(1 + t)^{-1} \geq 1 - t$ for $t \geq 0$ to obtain

$$\frac{r^2}{2(2r - 1 + (r - 1)^2 \lambda)} \geq \frac{r}{4(1 + r\lambda/2)} \geq \frac{r}{4} - \frac{r^2 \lambda}{8} \quad (\lambda \geq 0).$$

$\square$

The following lemma extends the exact factorization and rank bounds in [14, equations (2.11)–(2.12)] to permutation assignments with triangular defects.

LEMMA 3.4. *Over $\mathbb{F}_q$, let $Z_x = \sum_{i \in J(x)} Y_i$, an $rn \times rn$ matrix. For every $x \in \mathbb{F}_q^5$,*

$$\text{rank } Z_x \leq n + n \sum_{\substack{i \in J(x) \\ u, v \in [r]}} \delta_{x, i, u, v}. \quad (14)$$

PROOF. Permutation matrices P, Q differing on k points have at most k nonzero columns in $P - Q$. Thus $\text{rank}(P - Q) \leq nd_H(P, Q)$ over every field. Bi-invariance gives

$$\text{rank}(A(i, u, v) - L(x, u, \alpha_x(i) + v)^{-1}R(x, v, \beta_x(i) + u)) \leq n\delta_{x,i,u,v}.$$

Fix a point x . In Z_x , replace each permutation matrix by the corresponding product in the preceding display. For fixed u, v , the pairs $(\alpha_x(i) + v, \beta_x(i) + u)$, with $i \in J(x)$, traverse the whole Cartesian product once. The resulting matrix is

$$\begin{pmatrix} \sum_j L(x, 0, j)^{-1} \\ \vdots \\ \sum_j L(x, r-1, j)^{-1} \end{pmatrix} (\sum_k R(x, 0, k) \quad \cdots \quad \sum_k R(x, r-1, k)). \quad (15)$$

Both factors have rank at most n , so their product has rank at most n . Adding the difference in one matrix block increases the rank by at most the rank of that difference. Subadditivity of rank now gives (14). $\square$

PROOF OF THEOREM 3.1. Recall that $H_x \in M_I(\mathbb{F}_q)$ and $Y_i, Z_x \in M_{rn}(\mathbb{F}_q)$. We use the identifications

$$M_I(\mathbb{F}_q) \otimes_{\mathbb{F}_q} M_{rn}(\mathbb{F}_q) \cong M_I(M_{rn}(\mathbb{F}_q)) \cong M_{s^5 rn}(\mathbb{F}_q).$$

Thus the matrices have $s^5 \times s^5$ blocks, each of size $rn \times rn$. The symbol $\otimes$ denotes the Kronecker product: the (j, k) block of $H_x \otimes Z_x$ is $(H_x)_{jk} Z_x$. We write $\bigoplus_{i \in I} Y_i$ for the block diagonal matrix with diagonal blocks Y_i and zero off-diagonal blocks. From (5), we obtain the tensor identity of [14, equation (2.14)]:

$$-\bigoplus_{i \in I} Y_i = \sum_{x \in \mathbb{F}_q^5} H_x \otimes Z_x. \quad (16)$$

Indeed, expanding Z_x and changing the order of summation makes the coefficient of Y_i equal to $-E_{ii}$ for $i \in I$, and zero for $i \in J \setminus I$. The rank on the left of (16) is the sum of the individual ranks. Since $\text{rank } H_x \leq 1$ by Lemma 2.1, each summand on the right satisfies $\text{rank}(H_x \otimes Z_x) = \text{rank } H_x \text{ rank } Z_x \leq \text{rank } Z_x$. We apply Lemma 3.4 at every point x . Each triangular face $\tau = (x, i, u, v)$ has a unique associated point, so its defect is counted exactly once. Rank subadditivity therefore gives

$$\sum_{i \in I} \text{rank}_{\mathbb{F}_q} Y_i \leq \sum_{x \in \mathbb{F}_q^5} \text{rank}_{\mathbb{F}_q} Z_x \leq \sum_{x \in \mathbb{F}_q^5} \left(n + n \sum_{\substack{i \in J(x) \\ u, v \in [r]}} \delta_{x,i,u,v} \right) = n \left(q^5 + \sum_{\tau} \delta_{\tau} \right).$$

Using the second bound of Lemma 3.3 and cancelling the positive real number n , we obtain

$$\frac{rs^5}{4} - q^5 \leq \sum_{\tau} \delta_{\tau} + \frac{r^2}{8} \sum_{i \in I} \lambda_i.$$

Since $q < 10^{23}$ and $rs^5 = 59049 \cdot 10^{111} > 5 \cdot 10^{115}$, we have $rs^5/4 - q^5 > q^5/4 > q^5/5$. This proves (11). $\square$

4. The negatively curved complex

We equip every face of the complex K from Section 3 with the hyperbolic triangle of curvature -1 with angles

$$\theta_O = \pi/2, \quad \theta_V = \theta_W = \pi/5. \quad (17)$$

The angle sum is $9\pi/10 < \pi$, so this triangle exists. Assign to all L - and R -edges the equal side lengths opposite θ_W and θ_V , respectively, and to all A -edges the side length opposite θ_O . This includes edges contained in no face. All gluings are therefore isometric.

An edge germ at a vertex is an incident edge together with that endpoint. The angular link has one vertex for each edge germ and one edge for each face corner. The length of a link edge is the angle at the corresponding corner. A path in a graph is reduced if it has no immediate edge reversal. A reduced closed path is cyclically reduced if the last edge is not the inverse of the first edge. We set the distance between directions in different link components equal to infinity.

We adapt the link argument of [14, Lemma 2.3] to the present girth bound and angles.

LEMMA 4.1. *Every nonempty cyclically reduced closed path in any vertex link has length at least 2π . At either V or W , two different A -directions with the same index i have distance at least 2π , or belong to different components.*

PROOF. At O , the link is simple and bipartite: a corner between $L_{x,u,j}$ and $R_{x,v,k}$ determines $i \in J(x)$ uniquely by $\alpha_x(i) = j - v$ and $\beta_x(i) = k - u$. Every nonempty cyclically reduced closed path therefore has at least four edges and length at least $4\theta_O = 2\pi$.

We map the link at V to Γ by sending the direction $A_{i,u,v}$ to the line vertex i and the direction $L_{x,u,j}$ to the point x . Each corner maps to the corresponding incidence edge. We claim that this is an immersion, that is, the map is injective on incident edges at every vertex. Indeed, at $A_{i,u,v}$, a fixed point z determines at most one corner, namely that of the face indexed by (z, i, u, v) if $i \in J(z)$. At $L_{x,u,j}$, a fixed index i determines $v = j - \alpha_x(i)$ modulo s_x . There is at most one such $v \in [r]$, since $s_x \geq r$. At W , the same argument uses $u = k - \beta_x(i)$ modulo t_x and $t_x \geq r$.

Under these immersions, both a nonempty cyclically reduced closed link path and a reduced path joining distinct A -directions with the same index i map to nonempty closed walks in Γ with no immediate reversal at an internal vertex. Every such walk contains a simple cycle and hence has at least ten edges. Since link edges at V and W have length $\pi/5$, both assertions follow. $\square$

PROPOSITION 4.2. *The universal cover $\tilde{K}$ is a proper CAT(-1) space. The group $G = \pi_1(K, O)$ is torsion-free and word-hyperbolic, and K is a finite two-dimensional classifying space for G .*

PROOF. We first subdivide the complex so that cells meet in single faces. Subdivide each edge at its midpoint and each face by joining an interior point to its vertices and edge midpoints by geodesics. Since the closed cells embed and each edge and face has a distinct new vertex, the subdivision is a finite simplicial complex with the same intrinsic metric. The angular link at a new face-interior vertex is a circle of length 2π . At an edge-interior vertex it consists of length- π paths between the two edge directions.

The original angular links have the same metric after subdivision. By Lemma 4.1, their injective circuits have length at least 2π , and all circuits in the new links have length at least 2π . It follows from the metric graph criterion [3, Lemma II.5.6] and the link condition [3, Theorem II.5.2] that K is locally CAT(-1).

The lifted complex is locally finite and has finitely many shapes. Its length metric is complete by [3, Theorem I.7.13]. By local compactness and [3, Proposition I.3.7], it is proper and geodesic. The Cartan–Hadamard theorem [3, Theorem II.4.1] now implies that the universal cover $\tilde{K}$ is globally CAT(-1). Deck transformations act freely, properly and cocompactly by isometries. A finite group of isometries of a complete CAT(0) space fixes a point [3, Corollary II.2.8]. Since the deck action is free, no nonidentity deck transformation has finite order. The space is hyperbolic [3, Chapter III.H], so the Švarc–Milnor lemma [3, Proposition I.8.19] implies that G is word-hyperbolic. Since $\tilde{K}$ is contractible, K is a classifying space. $\square$

It remains to show that the rectangular words studied before are all non-trivial in G . This is the content of the following lemma. We adapt the argument following [14, equation (2.6)] to the CAT(-1) metric constructed above.

LEMMA 4.3. *For $i \in I$, $u \neq u'$ and $v \neq v'$, the element $w_{i;u,u',v,v'}$ is nonidentity in G .*

PROOF. The edge path $p = A_{i,u,v}A_{i,u',v}^{-1}A_{i,u',v'}A_{i,u,v'}^{-1}$ is a loop based at V , and $w_{i;u,u',v,v'}$ is the class of the loop $c_V p c_V^{-1}$ based at O . At every turn of p , including the closing turn, the two A -directions are distinct and have the same index i . By Lemma 4.1, their link distance is greater than π . It follows that p is a positive-length closed local geodesic, by the link description of angles [3, Chapter I.7]. If this loop were nullhomotopic, it would lift to a positive-length closed local geodesic in $\tilde{K}$. Every local geodesic in a CAT(-1) space is a geodesic [3, Proposition II.1.4]. This is impossible for a positive-length closed path. $\square$

5. Proof of the main result

Let H be a sofic group and let $\eta > 0$. Given finitely many words in a finite alphabet of elements of H and their inverses, choose a finite set $F \subset H$ containing the alphabet, its inverses, all word prefixes,

and 1. Let $\phi : F \rightarrow \text{Sym}(n)$ satisfy the inequalities in the definition of soficity from the introduction. For a word of length $\ell \geq 1$ with k inverse letters, evaluate it using the assigned permutations of the positive letters. Its distance from the image under ϕ of the corresponding group element is at most $(\ell - 1 + k)\eta$. Indeed, each inverse substitution changes the evaluated permutation by less than η , since $\phi(a)\phi(a^{-1})$ is within η of id . Each successive prefix multiplication introduces an error of less than η . The asserted bound follows from bi-invariance and the triangle inequality.

PROOF OF THEOREM A. The geometric assertions follow from Proposition 4.2. By Lemma 4.3, all rectangular elements are nonidentity. Assume that G is sofic. Apply the preceding estimate in G to all triangular and rectangular words, with the elements g_e as the alphabet. Assign $\phi(g_e)$ to each edge symbol. Each triangular word has length three, one inverse letter, and group value 1. Hence $\delta_\tau < 3\eta$. Each rectangular word has length four and two inverse letters, so its evaluated permutation is within 5η of $\phi(w)$. Since $w \neq 1$, the latter permutation has fixed-point fraction less than η . Thus the evaluated rectangle has fixed-point fraction less than 6η , and $\lambda_i < 6\eta$. These estimates also hold when some alphabet elements equal 1.

Each point x indexes $r^2|J(x)|$ triangular faces. By the degree count in Section 2, there are $r^2((q-1)q^5 - s^5)$ triangle indices in total. The data are fixed independently of n and η . Applying (11) from Theorem 3.1 to the permutations just constructed, and using the s^5 distinguished lines, gives

$$\frac{q^5}{5} \leq \sum_{\tau} \delta_{\tau} + \frac{r^2}{8} \sum_{i \in I} \lambda_i \leq 3r^2((q-1)q^5 - s^5)\eta + \frac{r^2}{8} \cdot 6s^5\eta \leq 3r^2q^6\eta.$$

Choosing $0 < \eta < 5 \cdot 10^{-37}$ gives a contradiction to our choices for q and r . $\square$

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