Notes on lattices in $\mathrm{SL}_3(\mathbb{R})$

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Contents

1 Introduction

The aim of these notes is to give a self-contained account of the classification of lattices in $\mathrm{SL}_3(\mathbb{R})$ up to commensurability, following the general strategy outlined by Margulis's theorems and the theory of algebraic groups. We will not give a proof of Margulis's theorems themselves, but we will use it as a key input in the classification. The main result is that every lattice in $\mathrm{SL}_3(\mathbb{R})$ is arithmetic, and the commensurability classes of lattices correspond bijectively to certain $\mathbb{Q}$-forms of SL_3, more precisely, to simply connected absolutely almost simple $\mathbb{Q}$-groups of type A_2 with real points isomorphic to $\mathrm{SL}_3(\mathbb{R})$, up to the diagram automorphism.

To cut a long story short the reader can refer right away to Section 8 to learn about four different constructions of lattices in $\mathrm{SL}_3(\mathbb{R})$, and then to Section 5 for the main classification result. The rest of the text is devoted to giving the necessary background and details for the proof that these four cases are the only that arise. In the course of the argument, we will need to recall some background on linear algebraic groups, central simple algebras, and Galois cohomology, which will allow us to give a self-contained account of the classification of $\mathbb{Q}$-forms of SL_3 and their arithmetic subgroups.

We start out by recalling the definitions of lattices and commensurability.

Definition 1.1. Let H be a locally compact group with Haar measure. A subgroup $\Gamma < H$ is a *lattice* if it is discrete and H/Γ has finite Haar volume. It is *uniform* if H/Γ is compact, and *non-uniform* otherwise.

Definition 1.2. Two subgroups $\Gamma_1, \Gamma_2 \leq G$ are *commensurable in* G if $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2. They are *abstractly commensurable* if some finite-index subgroup of Γ_1 is isomorphic to some finite-index subgroup of Γ_2.

The cornerstone of the theory of lattices in higher-rank Lie groups is the famous Margulis Superrigidity Theorem. We state it here only for lattices in $\mathrm{SL}_3(\mathbb{R})$, which is the case we are interested in. We are not going to give a proof of this theorem, but will use it as a black box input in the classification of lattices in $\mathrm{SL}_3(\mathbb{R})$.

Theorem 1.3 (Margulis)**.** *Let $\Gamma < \mathrm{SL}_3(\mathbb{R})$ be a lattice and let k be a local field of characteristic 0. Let $\rho : \Gamma \to \mathrm{GL}_n(k)$ be a group homomorphism. Then, either $\rho(\Gamma)$ is relatively compact in $\mathrm{GL}_n(k)$ or there exist a finite-index subgroup $\Gamma_0 < \Gamma$ and a continuous homomorphism $\widehat{\rho} : \mathrm{SL}_3(\mathbb{R}) \to \mathrm{GL}_n(k)$, such that $\rho = \widehat{\rho}|_{\Gamma_0}$.*

As a consequence, we will prove Margulis's arithmeticity theorem (Theorem 4.2) in the special case $\mathrm{SL}_3(\mathbb{R})$, which states that every lattice in $\mathrm{SL}_3(\mathbb{R})$ is arithmetic. All this holds in suitable form for irreducible lattices in semi-simple Lie groups of real rank at least 2. See [6, Ch. IX] or [9, Ch. 16].

The next sections are devoted to spell out what it means for a lattice in $\mathrm{SL}_3(\mathbb{R})$ to be arithmetic, and to give the necessary background on linear algebraic groups (Section 2), central simple algebras (Section 3), prove Margulis arithmeticity theorem (Section 4), and explain the use of Galois cohomology (Section 6), which will altogether allow us to give a self-contained account of the classification of lattices in $\mathrm{SL}_3(\mathbb{R})$ in Section 7.

The general classification of lattices in semi-simple groups in higher rank is more complicated than the A_2 case, and the A_2 case is already rich enough to illustrate the main ideas and techniques. The additional complication in the semi-simple case arises from the fact that the ambient algebraic group can be a product of several simple factors, and the classification of $\mathbb{Q}$-forms of such products is more involved. Moreover, in the semi-simple case one also has to deal with the possibility of non-trivial central isogenies between different $\mathbb{Q}$-forms, which can lead to additional subtleties in the classification. For a more general classification of lattices in higher rank, see [10].

Our starting point was the attempt to understand the details in Dave Morris Witte's notes [8], which give a sketch of the classification of lattices in $\mathrm{SL}_3(\mathbb{R})$, but do not spell out all the details in a way that was necessary for us to understand. We hope that these notes will be helpful for readers who want to understand the classification in more detail, and to see how the various pieces fit together. We found it very instructive to spell out the details of examples and to also include a proof of Margulis's arithmeticity theorem in the special case of $\mathrm{SL}_3(\mathbb{R})$.

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2 Linear algebraic groups

2.1 Definitions

In this section we recall the definitions of linear algebraic groups, arithmetic subgroups, and $\mathbb{Q}$-forms, which are the main players in the classification of lattices in $\mathrm{SL}_3(\mathbb{R})$.

Definition 2.1. A *linear algebraic group over* $\mathbb{Q}$ is an affine group scheme of finite type over $\mathbb{Q}$. Equivalently, it is a functor

$$G : (\text{commutative } \mathbb{Q}\text{-algebras}) \longrightarrow (\text{groups})$$

that is representable by a finitely generated Hopf $\mathbb{Q}$-algebra. For any $\mathbb{Q}$-algebra R, the associated group of R-points is written $G(R)$.

By standard structure results, any such G admits a faithful $\mathbb{Q}$-representation, so it can be realized as a Zariski-closed subgroup of some GL_n cut out by polynomial equations with coefficients in $\mathbb{Q}$. For definitions and basic properties, see [3, 7, 10].

Definition 2.2. Let k be a field and let G be a linear algebraic group over k. We say that G is *connected* if it is connected in the Zariski topology. References: [10, §2.1], [3, 7].

Definition 2.3. Let k be a field and let G be a connected linear algebraic group over k. Its *solvable radical* $\mathrm{Rad}(G)$ is the largest connected solvable normal k-subgroup. The group G is called *semi-simple* if $\mathrm{Rad}(G)$ is trivial. References: [10, §3.1], [3, 7].

Definition 2.4. Let k be a field, let $k \subseteq k'$ be a field extension, and let G be a linear algebraic group over k. The *base change* of G from k to k' is the k'-group

$$G_{k'} := G \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k').$$

On the level of functors of points one has, for every k'-algebra R,

$$G_{k'}(R) = G(R),$$

where R is viewed as a k-algebra via $k \to k' \to R$.

Definition 2.5. Let k be a field and let G be a connected semi-simple algebraic group over k. We say that G is *almost simple over* k if it has no proper non-trivial connected normal algebraic subgroups defined over k. We say that G is *absolutely almost simple* if, after base change to an algebraic closure $\overline{k}$, the group $G_{\overline{k}}$ is almost simple over $\overline{k}$. References: [10, §3.1], [3, 7].

Definition 2.6. Let G be a linear algebraic group over $\mathbb{Q}$. A subgroup $\Gamma \leq G(\mathbb{Q})$ is *arithmetic* if there exists a faithful $\mathbb{Q}$-embedding $G \hookrightarrow \mathrm{GL}_n$ such that Γ is commensurable with $G(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$.

Definition 2.7. Let G be a linear algebraic group over $\mathbb{Q}$. An *integral model* of G is a group scheme $\mathcal{G}$ over $\mathbb{Z}$ of finite type together with an isomorphism of $\mathbb{Q}$-groups $\mathcal{G}_{\mathbb{Q}} \cong G$. We write $\mathcal{G}(\mathbb{Z})$ for its group of integral points.

Lemma 2.8. *Let G be a linear algebraic group over $\mathbb{Q}$ and let $\Gamma \leq G(\mathbb{Q})$. Then Γ is arithmetic if and only if there exists an integral model $\mathcal{G}$ of G over $\mathbb{Z}$ such that Γ is commensurable in $G(\mathbb{Q})$ with $\mathcal{G}(\mathbb{Z})$.*

Proof sketch. Starting from a faithful embedding $G \hookrightarrow \mathrm{GL}_n$, one takes the schematic Zariski closure of G in $\mathrm{GL}_{n,\mathbb{Z}}$ to obtain an integral model $\mathcal{G}$ with $\mathcal{G}(\mathbb{Z}) = G(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$. Conversely, given an integral model $\mathcal{G}$, one can embed $\mathcal{G}$ into some $\mathrm{GL}_{n,\mathbb{Z}}$ and compare $\mathcal{G}(\mathbb{Z})$ with the corresponding matrix integral points. See [1, 9, 10]. □

Lemma 2.9. *Let G be a semi-simple algebraic group over $\mathbb{Q}$. Any two arithmetic subgroups of $G(\mathbb{Q})$ are commensurable.*

Proof. Using an integral model of G over $\mathbb{Z}$, any arithmetic subgroup is commensurable with $G(\mathbb{Z})$ for *some* choice of $\mathbb{Z}$-structure. Two $\mathbb{Z}$-structures yield commensurable groups of integral points. For details, see [9, §1.2–§1.4] or [10, Ch. 5]. □

Definition 2.10. Let H be an algebraic group over $\mathbb{R}$. A $\mathbb{Q}$-*form* of H is an algebraic group G over $\mathbb{Q}$ together with an isomorphism of $\mathbb{R}$-groups $G_{\mathbb{R}} \cong H$.

Definition 2.11. Let k be a field and let G be a connected linear algebraic group over k. A subgroup $B \leq G$ is called a *Borel subgroup* if, after base change to an algebraic closure $\overline{k}$, the group $B_{\overline{k}}$ is a maximal connected solvable closed subgroup of $G_{\overline{k}}$. We say that B is *defined over* k if it is a k-subgroup of G.

Definition 2.12. Let k be a field. A connected reductive linear algebraic group G over k is called *split over* k if it contains a Borel subgroup defined over k and a maximal torus $T \leq G$ defined over k that is k-isomorphic to $(\mathbb{G}_m)^r$ for some r. For example, SL_n is split over any field.

Definition 2.13. Let k be a field. A morphism $f\colon G \to H$ of algebraic groups over k is called an *isogeny* if f is surjective and $\ker(f)$ is finite. Two connected algebraic groups G, H over k are called *isogenous* if there exists an isogeny $G \to H$.

Definition 2.14. Let k be a field and let G be a connected absolutely almost simple algebraic group over k. We say that G is *of type* A_2 if, after base change to an algebraic closure $\overline{k}$, the absolute root system of $G_{\overline{k}}$ is of Dynkin type A_2 (equivalently: $G_{\overline{k}}$ is isogenous to SL_3).

Remark 2.15. The Dynkin diagram of type A_2 has a non-trivial symmetry that swaps the two simple roots. For the split group SL_3 this yields an outer $\mathbb{Q}$-automorphism, realized on matrices by

$$\theta(g) = (g^{-1})^{\mathsf{t}}.$$

When classifying lattices in $\mathrm{SL}_3(\mathbb{R})$ up to commensurability, we identify $\mathbb{Q}$-forms that differ by composing the chosen real identification $G(\mathbb{R}) \cong \mathrm{SL}_3(\mathbb{R})$ with θ.

Remark 2.16. Let $H = \mathrm{SL}_3$ viewed as an algebraic group over $\mathbb{R}$. If G is a $\mathbb{Q}$-*form* of H in the sense of Definition 2.10, then G is automatically simply connected, absolutely almost simple, and of type A_2. Conversely, if G is a simply connected absolutely almost simple $\mathbb{Q}$-group of type A_2 such that $G_{\mathbb{R}}$ is split, then choosing an $\mathbb{R}$-isomorphism $G_{\mathbb{R}} \cong \mathrm{SL}_3$ makes G into a $\mathbb{Q}$-form of SL_3. Note that being of type A_2 alone does *not* determine the real Lie group $G(\mathbb{R})$: for example, outer forms attached to an imaginary quadratic extension give real unitary groups $\mathrm{SU}(p, q)$. Thus, in this text, the "$\mathbb{Q}$-forms of SL_3" that occur are exactly those $\mathbb{Q}$-groups of type A_2 with $G(\mathbb{R}) \cong \mathrm{SL}_3(\mathbb{R})$ together with a choice of real identification, understood up to the diagram automorphism.

2.2 Arithmetic subgroups as lattices

Let G be a linear algebraic group over $\mathbb{Q}$. Then $G(\mathbb{R})$ carries a canonical structure of a locally compact Hausdorff topological group (in fact, a real Lie group). Concretely, choose any closed $\mathbb{Q}$-embedding $G \hookrightarrow \mathrm{GL}_n$. Identifying $\mathrm{GL}_n(\mathbb{R})$ with an open subset of $\mathbb{R}^{n^2}$, we give $G(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{R})$ the subspace topology. This topology is independent of the chosen faithful embedding and agrees with the usual real-analytic/Lie group topology on $\mathbb{R}$-points. In particular, $G(\mathbb{R})$ admits a Haar measure.

Theorem 2.17 (Borel–Harish-Chandra)**.** *Let G be a connected semi-simple algebraic group over $\mathbb{Q}$. Then every arithmetic subgroup $\Gamma \leq G(\mathbb{Q})$ is a lattice in $G(\mathbb{R})$.*

The original statement is in [1]; modern expositions include [9, 10]. The converse is false in general, but it holds for irreducible lattices in higher-rank Lie groups by Margulis's arithmeticity theorem; in particular, it holds for lattices in $\mathrm{SL}_3(\mathbb{R})$. So the basic strategy for classifying lattices in $\mathrm{SL}_3(\mathbb{R})$ up to commensurability is to classify the relevant $\mathbb{Q}$-forms of SL_3 and their arithmetic subgroups.

3 Central simple algebras

3.1 Definitions

Central simple algebras are the algebraic objects that underlie the classification of $\mathbb{Q}$-forms of SL_3.

Definition 3.1. Let k be a field. A finite-dimensional k-algebra A is called a *central simple algebra over k* if A is simple and its center is exactly k. Its *degree* is the integer n such that $\dim_k(A) = n^2$. For any k-algebra R, there is a *reduced norm* map $\mathrm{Nrd}\colon (A \otimes_k R)^\times \to R^\times$. The algebra A is *split* if $A \simeq M_n(k)$.

We will also need the notion of involutions of the second kind, which are the algebraic objects underlying the classification of outer $\mathbb{Q}$-forms of SL_3.

Definition 3.2. Let k be a field, let K/k be a quadratic field extension, and let B be a central simple K-algebra. An involution τ on B is called a *unitary involution* if it is an anti-automorphism of order 2 and its restriction to $K \subset B$ is the non-trivial element of $\mathrm{Gal}(K/k)$.

An involution seems like an additional piece of data but in fact the following lemma shows that a unitary involution is unique up to conjugation by an element fixed by the involution. The existence of a unitary involution on B is a non-trivial condition and will be discussed in more detail later, see Lemma 3.8 and Corollary 3.9.

Lemma 3.3. *Let k be a field, let K/k be a quadratic field extension, and let B be a central simple K-algebra. Assume that τ is a unitary involution on B.*

1. *If $b \in B^\times$ satisfies $\tau(b) = b$, then*

$$\tau_b := \mathrm{Int}(b) \circ \tau \qquad (\tau_b(x) = b\,\tau(x)\,b^{-1})$$

is again a unitary involution on B.

2. *Conversely, if τ' is any unitary involution on B, then there exists $b \in B^\times$ with $\tau(b) = b$ such that $\tau' = \tau_b$.*

Proof. (1) Since $\tau(b) = b$, we have $\tau_b^2 = \mathrm{Int}(b\,\tau(b)^{-1}) = \mathrm{id}$, so τ_b is an involution. Moreover $\mathrm{Int}(b)$ is K-linear, hence τ_b restricts to the same non-trivial automorphism of K/k as τ.

(2) The composition $\sigma := \tau' \circ \tau$ is a K-algebra automorphism of B, hence $\sigma = \mathrm{Int}(c)$ for some $c \in B^\times$ by Skolem–Noether. Thus $\tau' = \mathrm{Int}(c) \circ \tau$. The involution condition $\tau'^2 = \mathrm{id}$ implies that $u := c\,\tau(c)^{-1}$ lies in the center $K^\times$. Since $\tau(u) = u^{-1}$, Hilbert's Theorem 90 gives $t \in K^\times$ with $u = t^{-1}\tau(t)$. Then $b := tc$ satisfies $\tau(b) = b$, and $\tau' = \mathrm{Int}(b) \circ \tau$ because t is central. For a systematic treatment of involutions of the second kind, see [5, Ch. I, §2]. □

Lemma 3.4. *Let k be a field and let A be a central simple k-algebra of prime degree p. Then A is either split or a division algebra.*

Proof. By the Wedderburn structure theorem, $A \simeq M_r(D)$ for a central division k-algebra D. If $d = \deg(D)$ then $\deg(A) = rd$. Since $\deg(A) = p$ is prime, either $r = 1$ (so $A \simeq D$ is division) or $d = 1$ (so $D \simeq k$ and $A \simeq M_p(k)$ is split). □

3.2 The Brauer group and local invariants.

In this section we recall the definition of the Brauer group of a field and the classification of central simple algebras over number fields in terms of local invariants.

Definition 3.5. Let k be a field. Two central simple k-algebras A, B are *Brauer-equivalent* if there exist integers $m, n \geq 1$ such that $A \otimes_k M_m(k) \cong B \otimes_k M_n(k)$. The set of Brauer-equivalence classes forms an abelian group under tensor product, with neutral element the class of k (equivalently $M_r(k)$ for any r), and inverse given by the opposite algebra A^{op}. This group is called the *Brauer group* and is denoted $\mathrm{Br}(k)$. References: [13], [4].

For later use, it is convenient to recall the following more explicit description of $\mathrm{Br}(k)$ in terms of central division algebras.

Theorem 3.6 (Albert–Brauer–Hasse–Noether (ABHN))**.** *Let k be a number field. For each place v of k there is an invariant map*

$$\mathrm{inv}_v : \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$$

such that inv_v is an isomorphism for non-archimedean v (so $\mathrm{Br}(k_v) \cong \mathbb{Q}/\mathbb{Z}$), and for archimedean v one has $\mathrm{Br}(\mathbb{C}) = 0$ and $\mathrm{Br}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ (identified with $\frac{1}{2}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$ via inv_v). Moreover, the localization map yields an exact sequence

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \bigoplus_v \mathrm{Br}(k_v) \xrightarrow{\sum_v \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

For more details, see [13], [4].

Proposition 3.7. *Let k be a number field. Central division algebras A/k of degree 3 are classified by finite sets S of non-archimedean places of k together with a choice of $\varepsilon_v \in \{1, 2\} \subset \mathbb{Z}/3\mathbb{Z}$ for each $v \in S$, subject to the congruence*

$$\sum_{v \in S} \varepsilon_v \equiv 0 \pmod 3,$$

via

$$\mathrm{inv}_v(A) = \varepsilon_v/3 \ (v \in S), \qquad \mathrm{inv}_v(A) = 0 \ (v \notin S).$$

In particular, $\mathrm{inv}_v(A) = 0$ *for every archimedean place* v*.*

Proof sketch. For each non-archimedean place v of k one has $\mathrm{Br}(k_v) \cong \mathbb{Q}/\mathbb{Z}$ via inv_v. If A/k has degree 3, then for every v the local class $[A \otimes_k k_v] \in \mathrm{Br}(k_v)$ has exponent dividing 3; hence $\mathrm{inv}_v(A) \in \frac{1}{3}\mathbb{Z}/\mathbb{Z}$. At archimedean places one has $\mathrm{Br}(\mathbb{C}) = 0$ and $\mathrm{Br}(\mathbb{R})$ has no 3-torsion, so $\mathrm{inv}_v(A) = 0$. For a division algebra of degree 3, the local invariant is nonzero exactly at the (finite) set of ramified places, and at such a place it must equal 1/3 or 2/3. The global sum condition $\sum_v \mathrm{inv}_v(A) = 0$ is exactly ABHN, and conversely ABHN shows that any such finitely supported family occurs. See [4, §6.5]. □

3.3 Real quadratic fields and unitary involutions.

Let $K/\mathbb{Q}$ be a real quadratic field and let $\iota \in \mathrm{Gal}(K/\mathbb{Q})$ denote the non-trivial automorphism. For each place w of K one has a local invariant map $\mathrm{inv}_w \colon \mathrm{Br}(K_w) \to \mathbb{Q}/\mathbb{Z}$ and an ABHN exact sequence over K. In particular, a central simple K-algebra B of degree 3 is determined up to K-isomorphism by its family of local invariants $(\mathrm{inv}_w(B))_w$, which is finitely supported on finite places and satisfies $\sum_w \mathrm{inv}_w(B) = 0$.

Lemma 3.8. *Let* $K/\mathbb{Q}$ *be a quadratic extension with non-trivial automorphism* ι*, and let* B *be a central simple* K*-algebra. Then the following are equivalent:*

1. B *admits a* unitary involution *(i.e. an involution restricting to* ι *on* K*).*
2. $\iota([B]) = -[B]$ *in* $\mathrm{Br}(K)$ *or equivalently:* $\iota([B]) = [B^{\mathrm{op}}]$.
3. *For every place* w *of* K*, one has*

$$\mathrm{inv}_{\iota w}(B) = -\mathrm{inv}_w(B) \quad \textit{in } \mathbb{Q}/\mathbb{Z}.$$

Proof sketch with references. The equivalence of (2) and (3) is the compatibility of local invariants with Galois conjugation. Finally, (1)⇔(2) is a standard criterion for involutions of the second kind: a unitary involution on B exists if and only if $B \cong {}^{\iota}B^{\mathrm{op}}$ as K-algebras, which is exactly $\iota([B]) = -[B]$. See [5, Ch. I, §2] and [10, Ch. 6]. □

The action of ι on the set of places of K is as follows: if a rational prime p is inert or ramified in K, then there is a unique place $w \mid p$ and it is fixed by ι; if p is split in K, then there are two places $w, w' \mid p$ and $\iota w = w'$. Thus, we obtain the following explicit criterion for the existence of unitary involutions on degree-3 central simple algebras over real quadratic fields.

Corollary 3.9. *Let* $K/\mathbb{Q}$ *be real quadratic and let* B *be a central simple* K*-algebra of degree* 3*. Then* B *admits a unitary involution if and only if the following local condition holds:*

- *If a rational prime* p *is inert or ramified in* K *(so there is a unique place* $w \mid p$ *with* $\iota w = w$*), then* $\mathrm{inv}_w(B) = 0$*.*
- *If a rational prime* p *is split in* K *with places* $w, w' \mid p$*, then* $\mathrm{inv}_{w'}(B) = -\mathrm{inv}_w(B)$*.*

In other words, the local invariants vanish at every ι-fixed place and occur in opposite ι-pairs above split primes. In particular, degree-3 division algebras over K with unitary involution exist: one may choose finitely many split primes p and prescribe opposite local invariants at the two places of K over each such p.

Proof. Assume first that B admits a unitary involution. Then Lemma 3.8(3) gives $\mathrm{inv}_{\iota w}(B) = -\mathrm{inv}_w(B)$ for every place w. If p is inert or ramified, there is a unique place $w \mid p$ and it is fixed by ι, hence $\mathrm{inv}_w(B) = -\mathrm{inv}_w(B)$ and therefore $2\,\mathrm{inv}_w(B) = 0$. For degree 3 one has $\mathrm{inv}_w(B) \in \frac{1}{3}\mathbb{Z}/\mathbb{Z}$, so $2\,\mathrm{inv}_w(B) = 0$ forces $\mathrm{inv}_w(B) = 0$. If p is split with places w, w', then $w' = \iota w$ and Lemma 3.8(3) gives $\mathrm{inv}_{w'}(B) = -\mathrm{inv}_w(B)$.

Conversely, assume that $\mathrm{inv}_w(B) = 0$ for every ι-fixed place w, and that for every split p with places w, w' one has $\mathrm{inv}_{w'}(B) = -\mathrm{inv}_w(B)$. Then $\mathrm{inv}_{\iota w}(B) = -\mathrm{inv}_w(B)$ holds for all places w (trivially at ι-fixed places, and by assumption at split places), so B admits a unitary involution by Lemma 3.8.

The existence statement follows by the ABHN classification over K: assigning finitely many nonzero invariants at split places and their negatives at the conjugate places yields a valid global Brauer class. □

Let $K/\mathbb{Q}$ be a quadratic extension, let B be a central simple K-algebra, and let τ be a unitary involution on B. We write $\mathrm{SU}(B,\tau)$ for the associated (simply connected) special unitary $\mathbb{Q}$-group. Concretely, for any $\mathbb{Q}$-algebra R,

$$\mathrm{SU}(B,\tau)(R) = \{x \in (B \otimes_{\mathbb{Q}} R)^{\times} : (\tau \otimes \mathrm{id}_R)(x) \cdot x = 1 \text{ and } \mathrm{Nrd}(x) = 1\}.$$

For background see [5, 10].

3.4 Orders in central simple algebras

Let k be a number field with ring of integers $\mathcal{O}_k$, and let A be a finite-dimensional central simple k-algebra. Orders in A provide the integral structures needed to define arithmetic subgroups such as $\mathrm{SL}_1(\mathcal{O})$ and, in the presence of a unitary involution, unitary groups $\mathrm{SU}(\mathcal{O},\tau)$. Standard references are [11, 5].

Definition 3.10. An *$\mathcal{O}_k$-order* in A is an $\mathcal{O}_k$-subalgebra $\mathcal{O} \subset A$ that is a finitely generated $\mathcal{O}_k$-module and spans A over k, i.e.

$$k \cdot \mathcal{O} \;=\; A.$$

Equivalently, $\mathcal{O}$ is a full $\mathcal{O}_k$-lattice in the k-vector space A that is closed under multiplication and contains 1. For $k = \mathbb{Q}$ this is the usual notion of a $\mathbb{Z}$-order.

Definition 3.11. An $\mathcal{O}_k$-order $\mathcal{O} \subset A$ is *maximal* if it is not properly contained in any other $\mathcal{O}_k$-order in A.

Remark 3.12. Maximal orders always exist: every $\mathcal{O}_k$-order is contained in a maximal one. If v is a non-archimedean place of k, one may localize an order by $\mathcal{O}_v := \mathcal{O} \otimes_{\mathcal{O}_k} \mathcal{O}_{k_v} \subset A \otimes_k k_v$. For almost all v, the localized algebra $A \otimes_k k_v$ is split and a maximal order is isomorphic to $M_n(\mathcal{O}_{k_v})$. These facts are classical; see [11, Ch. 10–12].

Let K/k be a quadratic extension of number fields with ring of integers $\mathcal{O}_K$, let B be a central simple K-algebra, and let τ be a *unitary involution* on B (so $\tau|_K$ is the non-trivial element of $\mathrm{Gal}(K/k)$).

Definition 3.13. An $\mathcal{O}_K$-order $\mathcal{O} \subset B$ is called *τ-stable* if $\tau(\mathcal{O}) = \mathcal{O}$.

Lemma 3.14. *Let $\mathcal{O} \subset B$ be an $\mathcal{O}_K$-order. Then:*

1. *$\tau(\mathcal{O})$ is again an $\mathcal{O}_K$-order in B.*
2. *The intersection $\mathcal{O} \cap \tau(\mathcal{O})$ is a τ-stable $\mathcal{O}_K$-order.*

Proof. Since τ restricts on K to the non-trivial automorphism of K/k, it preserves $\mathcal{O}_K$. Moreover, τ is an additive bijection of B preserving 1 and multiplication (up to reversing order), so $\tau(\mathcal{O})$ is an $\mathcal{O}_K$-lattice in B closed under multiplication and spanning B over K, hence an $\mathcal{O}_K$-order. The intersection of two full $\mathcal{O}_K$-lattices in B is again a full $\mathcal{O}_K$-lattice, and it is clearly closed under multiplication and fixed by τ. □

Proposition 3.15. *There exists a τ-stable maximal $\mathcal{O}_K$-order in B.*

Proof sketch with references. This is standard in the arithmetic theory of algebras with involution: any τ-stable order is contained in a τ-stable maximal order. See [5, Ch. I, §2 and Ch. 26] and [11, Ch. 10–12]. □

4 Margulis superrigidity and arithmeticity

The goal of this section is to explain how Margulis's arithmeticity theorem for lattices in $\mathrm{SL}_3(\mathbb{R})$ (Theorem 4.2) can be deduced from Margulis's superrigidity theorem. We will focus entirely on the group $G = \mathrm{SL}_3(\mathbb{R})$, and we will present the argument as a sequence of concrete reductions.

4.1 Statement of superrigidity for $\mathrm{SL}_3(\mathbb{R})$

We use the following form of superrigidity, which is tailored to the needs of the arithmeticity argument. There are many equivalent formulations; the point here is that any linear representation of a lattice either has precompact image or essentially extends to the ambient Lie group. Recall that a local field of characteristic 0 is either $\mathbb{R}$, $\mathbb{C}$, or a finite extension of $\mathbb{Q}_p$.

Theorem 4.1 (Margulis)**.** *Let $\Gamma < \mathrm{SL}_3(\mathbb{R})$ be a lattice and let k be a local field of characteristic 0. Let $\rho : \Gamma \to \mathrm{GL}_n(k)$ be a group homomorphism. Assume that $\rho(\Gamma)$ is not relatively compact in $\mathrm{GL}_n(k)$. Then there exist*

- *a finite-index subgroup $\Gamma_0 < \Gamma$,*
- *a continuous homomorphism $\widehat{\rho} : \mathrm{SL}_3(\mathbb{R}) \to \mathrm{GL}_n(k)$,*

such that $\rho = \widehat{\rho}|_{\Gamma_0}$.

References for Theorem 4.1 include [6, Ch. VII–IX] and [9, Ch. 16].

4.2 Arithmeticity from superrigidity in the $\mathrm{SL}_3(\mathbb{R})$ case

We now explain how Theorem 4.1 implies the arithmeticity of lattices in $\mathrm{SL}_3(\mathbb{R})$.

Theorem 4.2 (Margulis)**.** *Every lattice $\Gamma < \mathrm{SL}_3(\mathbb{R})$ is arithmetic.*

Proof. We give a detailed outline of the standard argument, specialized to $\mathrm{SL}_3(\mathbb{R})$.

Step 1: Fix the inclusion $\Gamma \hookrightarrow \mathrm{SL}_3(\mathbb{R}) \subset \mathrm{SL}_3(\mathbb{C})$. Since lattices in Lie groups are finitely generated, we may choose a finite generating set $S \subset \Gamma$. Let $K \subset \mathbb{R}$ be the subfield generated over $\mathbb{Q}$ by the matrix entries of the elements of S. Then K is a finitely generated field extension of $\mathbb{Q}$, and every element of Γ has matrix entries in K. By the structure theory of finitely generated fields, K is a finite extension of a purely transcendental extension $\mathbb{Q}(t_1, \ldots, t_r)$. For every field embedding $\sigma : K \hookrightarrow \mathbb{C}$ we obtain a conjugate representation

$$\rho_\sigma : \Gamma \longrightarrow \mathrm{SL}_3(\mathbb{C}), \qquad \rho_\sigma(\gamma) = \big(\sigma(\gamma_{ij})\big)_{i,j}.$$

We apply Theorem 4.1 to the homomorphism ρ_σ with $k = \mathbb{C}$. Thus, for each σ, either

(a) the image $\rho_\sigma(\Gamma)$ is relatively compact in $\mathrm{SL}_3(\mathbb{C})$, or

(b) after passing to finite index, ρ_σ extends to a continuous homomorphism $\widehat{\rho}_\sigma : \mathrm{SL}_3(\mathbb{R}) \to \mathrm{SL}_3(\mathbb{C})$.

In case (b), the differential $d\widehat{\rho}_\sigma : \mathfrak{sl}_3(\mathbb{R}) \to \mathfrak{sl}_3(\mathbb{C})$ is a nonzero Lie algebra homomorphism, hence injective. Moreover, because the target group is $\mathrm{SL}_3(\mathbb{C})$, the only nontrivial connected real Lie subgroups of $\mathrm{SL}_3(\mathbb{C})$ locally isomorphic to $\mathrm{SL}_3(\mathbb{R})$ are (up to conjugacy) the standard real points $\mathrm{SL}_3(\mathbb{R}) \subset \mathrm{SL}_3(\mathbb{C})$ and their images under the diagram automorphism. Equivalently, in case (b) the representation $\widehat{\rho}_\sigma$ is, up to conjugacy and diagram automorphism, the inclusion $\mathrm{SL}_3(\mathbb{R}) \hookrightarrow \mathrm{SL}_3(\mathbb{C})$.

Step 2: force K to be a number field. We claim that $K/\mathbb{Q}$ is algebraic and hence finite, since K is finitely generated. Suppose not. Then K contains a purely transcendental subextension $\mathbb{Q}(t_1, \ldots, t_r)$ of positive transcendence degree r, so there exists an element $x = t_1 \in K$ whose values $|\sigma(x)|$ under embeddings $\sigma : K \hookrightarrow \mathbb{C}$ are unbounded. We are now going to define the *trace field* of Γ. Since Γ is Zariski dense in SL_3 (Borel density), the inclusion $\Gamma \hookrightarrow \mathrm{SL}_3(\mathbb{R})$ is absolutely irreducible. In this situation one may replace the field of definition K by the subfield generated by the traces $\{\mathrm{tr}(\gamma) : \gamma \in \Gamma\}$: the matrix entries of a finite generating set are algebraic over this trace field. More precisely, fix generators $S = \{\gamma_1, \ldots, \gamma_m\}$ and let

$$F := \mathbb{Q}\big(\mathrm{tr}(w(\gamma_1, \ldots, \gamma_m)) : \ w \text{ a word}\big) \subset \mathbb{R}$$

be the corresponding trace field (so in particular $\mathrm{tr}(\gamma_i)$ and $\mathrm{tr}(\gamma_i^2)$ lie in F). There is a general result (Procesi's theorem / trace identities for matrix invariants in characteristic 0) saying that, for an absolutely irreducible representation, the coordinate ring generated by the matrix entries is integral over the *trace algebra* generated by these traces. Equivalently, each matrix coefficient of the generators satisfies a nontrivial polynomial equation with coefficients in F, hence is algebraic over F. Hence, if K is not algebraic over $\mathbb{Q}$, we can (after replacing x) assume that

$$x = \mathrm{tr}(\gamma) \quad \text{for some fixed } \gamma \in \Gamma,$$

and there are embeddings $\sigma_n : K \hookrightarrow \mathbb{C}$ with $|\sigma_n(x)| \to \infty$.

For such σ_n we have

$$\mathrm{tr}\big(\rho_{\sigma_n}(\gamma)\big) = \sigma_n\big(\mathrm{tr}(\gamma)\big) = \sigma_n(x),$$

so in particular the set of complex numbers $\{\mathrm{tr}(\rho_{\sigma_n}(\gamma))\}_n$ is unbounded and we find $n \in \mathbb{N}$ such that $|\mathrm{tr}(\rho_{\sigma_n}(\gamma))| > 3$. If $\rho_{\sigma_n}(\Gamma)$ were relatively compact, then its closure would be a compact subgroup of $\mathrm{SL}_3(\mathbb{C})$. Every compact subgroup of $\mathrm{GL}_3(\mathbb{C})$ is conjugate into $\mathrm{U}(3)$, hence every element of it has all eigenvalues on the unit circle and therefore $|\mathrm{tr}| \le 3$. Since the trace is invariant under conjugation, this bound holds for elements of any compact subgroup of $\mathrm{SL}_3(\mathbb{C})$. Thus $|\mathrm{tr}(\rho_{\sigma_n}(\gamma))| > 3$ forces $\rho_{\sigma_n}(\Gamma)$ to be *not* relatively compact. By superrigidity (Step 1), each such ρ_σ must then fall into case (b), hence extend to $\mathrm{SL}_3(\mathbb{R})$. But now the classification in Step 1 gives a contradiction at the level of traces: if ρ_{σ_n} extends, then up to conjugacy and the diagram automorphism we have

$$\rho_{\sigma_n}(\gamma) = h_n\,\gamma\,h_n^{-1} \quad \text{or} \quad \rho_{\sigma_n}(\gamma) = h_n\,(\gamma^{-1})^{\mathrm{t}}\,h_n^{-1},$$

with $h_n \in \mathrm{SL}_3(\mathbb{C})$. Taking traces (which are invariant under conjugation) shows

$$\mathrm{tr}\big(\rho_{\sigma_n}(\gamma)\big) \in \{\mathrm{tr}(\gamma),\ \mathrm{tr}(\gamma^{-1})\},$$

which is a *finite* set, hence bounded. This contradicts $|\mathrm{tr}(\rho_{\sigma_n}(\gamma))| = |\sigma_n(x)| \to \infty$. This contradiction shows that K is algebraic over $\mathbb{Q}$, hence a number field.

Step 3: the diagonal embedding. Let v range over the archimedean places of K. For each v corresponding to an embedding $\sigma_v : K \hookrightarrow \mathbb{C}$, we obtain a homomorphism $\rho_v = \rho_{\sigma_v} : \Gamma \to \mathrm{SL}_3(\mathbb{C})$. By Step 1, for all but one archimedean place (namely the one coming from the original embedding $K \subset \mathbb{R}$), the image is relatively compact. Indeed, we may (and do) take K to be the trace field $K = \mathbb{Q}(\mathrm{tr}(\gamma) : \gamma \in \Gamma)$. Let v be an archimedean place of K with embedding $\sigma_v : K \hookrightarrow \mathbb{C}$. If $\rho_v(\Gamma)$ were not relatively compact, then by superrigidity (Step 1) it would (after passing to finite index) extend to a continuous homomorphism $\widehat{\rho}_v : \mathrm{SL}_3(\mathbb{R}) \to \mathrm{SL}_3(\mathbb{C})$. By Step 1, up to conjugacy and the diagram automorphism, $\widehat{\rho}_v$ is the standard inclusion $\mathrm{SL}_3(\mathbb{R}) \hookrightarrow \mathrm{SL}_3(\mathbb{C})$. In particular, for every $\gamma \in \Gamma$ we would have

$$\sigma_v(\mathrm{tr}(\gamma)) = \mathrm{tr}(\rho_v(\gamma)) \in \{\mathrm{tr}(\gamma),\ \mathrm{tr}(\gamma^{-1})\}.$$

The right-hand side consists of real numbers (since $\gamma \in \mathrm{SL}_3(\mathbb{R})$), so $\sigma_v(\mathrm{tr}(\gamma)) \in \mathbb{R}$ for all γ. Because the traces generate K, this forces $\sigma_v(K) \subset \mathbb{R}$, i.e. v must be a *real* place. Moreover, the displayed trace constraint (holding for all γ) forces σ_v to agree with the original real embedding on the trace field, so v is exactly the distinguished archimedean place coming from $K \subset \mathbb{R}$.

Therefore every other archimedean place must fall into case (a), i.e. have relatively compact image. Equivalently, after choosing a suitable K-form $\mathbf{H}$ of type A_2 whose real points at the distinguished place are isomorphic to $\mathrm{SL}_3(\mathbb{R})$, the group Γ embeds diagonally into

$$\prod_{v|\infty} \mathbf{H}(K_v) \cong \mathrm{SL}_3(\mathbb{R}) \times (\text{compact}).$$

At this point one invokes the reduction theory of arithmetic groups, in the form of the Borel–Harish-Chandra theorem for S-arithmetic subgroups. Concretely, choose a K-isomorphism $\mathbf{H} \otimes_K \overline{K} \cong \mathrm{SL}_3 \otimes_K \overline{K}$ and view Γ as a subgroup of $\mathbf{H}(K)$ via the chosen K-structure. Let $S \subset V_f(K)$ be the finite set of non-archimedean places where denominators occur in a fixed finite generating set of Γ: equivalently, if $R \subset K$ denotes the finitely generated $\mathbb{Z}$-algebra generated by all matrix coefficients of the chosen generators, then $R \subset \mathcal{O}_{K,S}$, where

$$\mathcal{O}_{K,S} := \{x \in K :\ v_w(x) \ge 0 \text{ for all } w \notin S\}$$

is the ring of S-integers. After conjugating inside $\mathrm{SL}_3(K)$ (equivalently, choosing a suitable $\mathcal{O}_{K,S}$-lattice), this gives an inclusion

$$\Gamma \subseteq \mathbf{H}(\mathcal{O}_{K,S})$$

up to commensurability.

The Borel–Harish-Chandra theorem (in its S-arithmetic form) says that $\mathbf{H}(\mathcal{O}_{K,S})$ embeds diagonally as a lattice in

$$\prod_{v|\infty} \mathbf{H}(K_v) \times \prod_{w\in S} \mathbf{H}(K_w).$$

At this point one should be careful: although all archimedean factors except the distinguished real one are compact, the non-archimedean factors $\prod_{w\in S} \mathbf{H}(K_w)$ are typically non-compact when $S \neq \varnothing$. Accordingly, the projection of the S-arithmetic lattice $\mathbf{H}(\mathcal{O}_{K,S})$ to the real factor $\mathbf{H}(K_{v_0}) \cong \mathrm{SL}_3(\mathbb{R})$ need not be discrete (compare $\mathrm{SL}_3(\mathbb{Z}[1/p]) \subset \mathrm{SL}_3(\mathbb{R})$). The point of the next step is to control the non-archimedean components: Step 3 shows that no denominators are needed at finite places, i.e. one may take $S = \varnothing$. Once $S = \varnothing$, Borel–Harish-Chandra gives a lattice in the purely archimedean product, and then (since the other archimedean factors are compact) the projection to $\mathbf{H}(K_{v_0}) \cong \mathrm{SL}_3(\mathbb{R})$ is indeed a lattice of finite covolume.

Step 3: non-archimedean places. The preceding step produces an S-arithmetic group a priori. To upgrade to an honest arithmetic group (i.e. to show that one may take $S = \varnothing$), one uses superrigidity again, this time over non-archimedean local fields.

Let w be a non-archimedean place of K and consider the homomorphism

$$\rho_w : \Gamma \longrightarrow \mathrm{SL}_3(K_w)$$

obtained by applying the embedding $K \hookrightarrow K_w$ to matrix coefficients (equivalently: base-change the K-structure). We claim that $\rho_w(\Gamma)$ is relatively compact in $\mathrm{SL}_3(K_w)$. If not, then Theorem 4.1 applied with the local field $k = K_w$ would imply that (after passing to finite index) ρ_w extends to a continuous homomorphism

$$\widehat{\rho}_w : \mathrm{SL}_3(\mathbb{R}) \to \mathrm{SL}_3(K_w).$$

But $\mathrm{SL}_3(\mathbb{R})$ is connected while $\mathrm{SL}_3(K_w)$ is totally disconnected, so the image of any continuous homomorphism $\mathrm{SL}_3(\mathbb{R}) \to \mathrm{SL}_3(K_w)$ is relatively compact. This contradicts the assumption that $\rho_w(\Gamma)$ is not relatively compact. Hence $\rho_w(\Gamma)$ is relatively compact for every non-archimedean w.

Finally, a standard compactness criterion in p-adic linear groups says that a relatively compact subgroup of $\mathrm{GL}_n(K_w)$ preserves an $\mathcal{O}_w$-lattice in K_w^n, so after conjugation it is contained in $\mathrm{GL}_n(\mathcal{O}_w)$ (and similarly in $\mathrm{SL}_n(\mathcal{O}_w)$). Applied to $\rho_w(\Gamma)$ for all w, this shows that no denominators are needed at finite places, i.e. one can take $S = \varnothing$. Thus the S-arithmetic subgroup from Step 2 is in fact arithmetic.

Step 4: descent to $\mathbb{Q}$ and removal of compact factors. Let $\mathbf{G} := \mathrm{Res}_{K/\mathbb{Q}}(\mathbf{H})$. Then $\mathbf{G}$ is a connected semisimple $\mathbb{Q}$-group and

$$\mathbf{G}(\mathbb{R}) \cong \prod_{v|\infty} \mathbf{H}(K_v) \cong \mathrm{SL}_3(\mathbb{R}) \times (\text{compact}).$$

The arithmetic subgroup constructed in Step 2 is an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$. Projecting to the noncompact $\mathrm{SL}_3(\mathbb{R})$ factor gives a lattice commensurable with Γ.

Finally, since $\mathrm{SL}_3(\mathbb{R})$ has trivial center and no nontrivial compact normal subgroups, one can replace $\mathbf{G}$ by the (unique) almost $\mathbb{Q}$-simple factor whose real points map onto the $\mathrm{SL}_3(\mathbb{R})$-factor with finite kernel. Taking the simply connected cover of this factor yields the stated $\mathbb{Q}$-group in the theorem.

This completes the derivation of arithmeticity from superrigidity in the $\mathrm{SL}_3(\mathbb{R})$ case. For a full treatment of the steps above, see [6, Ch. IX] or [9, Ch. 16]. □

In particular, Theorem 4.2 proves Theorem 4.2.

Remark 4.3. The argument above is the standard "superrigidity ⇒ arithmeticity" route. In the present notes we ultimately classify the resulting $\mathbb{Q}$-forms of type A_2 explicitly (Section 6), so the abstract $\mathbb{Q}$-group produced by restriction of scalars can be replaced by a $\mathbb{Q}$-form of SL_3 with real points exactly $\mathrm{SL}_3(\mathbb{R})$.

4.3 Commensurators and the ambient $\mathbb{Q}$-form

In later sections we will use commensurators to see that the commensurability class of a lattice determines the ambient $\mathbb{Q}$-group.

Definition 4.4. Let H be a group and $\Gamma < H$. The *commensurator* of Γ in H is

$$\mathrm{Comm}_H(\Gamma) := \{h \in H : h\Gamma h^{-1} \text{ is commensurable with } \Gamma\}.$$

Theorem 4.5 (Margulis)**.** *Let $\Gamma < \mathrm{SL}_3(\mathbb{R})$ be an arithmetic lattice. Let $G/\mathbb{Q}$ be the simply connected absolutely almost simple $\mathbb{Q}$-group with $G(\mathbb{R}) \cong \mathrm{SL}_3(\mathbb{R})$ such that Γ is commensurable with the image of an arithmetic subgroup of $G(\mathbb{Q})$. Then, up to finite index,*

$$\mathrm{Comm}_{\mathrm{SL}_3(\mathbb{R})}(\Gamma) = \textit{image of } G(\mathbb{Q}) \textit{ in } \mathrm{SL}_3(\mathbb{R}).$$

In particular, the commensurability class of Γ determines the $\mathbb{Q}$-isomorphism class of G up to the diagram automorphism of type A_2.

Proof for $\mathrm{SL}_3(\mathbb{R})$*.* We give a proof in the $\mathrm{SL}_3(\mathbb{R})$ setting that cleanly separates the formal reductions from the deep inputs.

Step 1: reduce to an arithmetic lattice coming from a fixed $\mathbb{Q}$-group. By assumption, there is a simply connected absolutely almost simple $\mathbb{Q}$-group G with $G(\mathbb{R}) \cong \mathrm{SL}_3(\mathbb{R})$ and an arithmetic subgroup $\Lambda < G(\mathbb{Q})$ such that (after identifying $G(\mathbb{R})$ with $\mathrm{SL}_3(\mathbb{R})$) the groups Γ and Λ are commensurable. Since commensurable subgroups have the same commensurator, we may replace Γ by Λ.

Step 2: the easy inclusion $G(\mathbb{Q}) \subseteq \mathrm{Comm}(\Gamma)$. Let $g \in G(\mathbb{Q})$. Then $g\Gamma g^{-1}$ is again an arithmetic subgroup of $G(\mathbb{Q})$. By Lemma 2.9, any two arithmetic subgroups of $G(\mathbb{Q})$ are commensurable, hence $g\Gamma g^{-1}$ is commensurable with Γ. Therefore $g \in \mathrm{Comm}_{G(\mathbb{R})}(\Gamma)$. Transporting across the fixed identification $G(\mathbb{R}) \cong \mathrm{SL}_3(\mathbb{R})$ gives the desired inclusion in $\mathrm{SL}_3(\mathbb{R})$.

Step 3: commensurator theorem and the reverse inclusion. Because $\mathrm{SL}_3(\mathbb{R})$ is connected, simple, and of real rank 2, every lattice in $\mathrm{SL}_3(\mathbb{R})$ is irreducible. Margulis's commensurator theorem for irreducible lattices in higher rank (applied to $\mathrm{SL}_3(\mathbb{R})$) says:

Either $\mathrm{Comm}_{\mathrm{SL}_3(\mathbb{R})}(\Gamma)$ is discrete and contains Γ with finite index, or it is dense in $\mathrm{SL}_3(\mathbb{R})$. Moreover, the dense case occurs if and only if Γ is arithmetic.

Since Γ is arithmetic by assumption, the theorem implies that $\mathrm{Comm}_{\mathrm{SL}_3(\mathbb{R})}(\Gamma)$ is dense.

To identify the commensurator more precisely, one uses the arithmetic refinement of the commensurator theorem: if Γ is commensurable with an arithmetic subgroup of $G(\mathbb{Q})$, then the commensurator is (up to finite index) exactly the image of $G(\mathbb{Q})$ in $G(\mathbb{R}) \cong \mathrm{SL}_3(\mathbb{R})$. Combined with Step 2, this gives the reverse inclusion up to finite index and proves the displayed equality.

Finally, the "in particular" statement follows because the $\mathbb{Q}$-isomorphism class of a simply connected absolutely almost simple $\mathbb{Q}$-group of type A_2 is determined by its group of $\mathbb{Q}$-points inside $\mathrm{SL}_3(\mathbb{R})$, up to composing the fixed identification $G(\mathbb{R}) \cong \mathrm{SL}_3(\mathbb{R})$ with the diagram automorphism.

For detailed proofs of both the commensurator theorem and its arithmetic refinement, see [6, Ch. IX] or [9, Ch. 16]. □

5 Statement of main theorem

We are now ready to spell out the classification of lattices in $\mathrm{SL}_3(\mathbb{R})$ up to commensurability, which is the main result of these notes.

Theorem 5.1. *Let $\Gamma < \mathrm{SL}_3(\mathbb{R})$ be a lattice. Then:*

1. *Γ is arithmetic: there exists a simply connected, absolutely almost simple $\mathbb{Q}$-group G with $G(\mathbb{R}) \cong \mathrm{SL}_3(\mathbb{R})$ such that Γ is commensurable in $\mathrm{SL}_3(\mathbb{R})$ with the image of an arithmetic subgroup of $G(\mathbb{Q})$.*

2. *Every such G is up to $\mathbb{Q}$-isomorphism of exactly one of the following two kinds:*

   (a) *Inner $\mathbb{Q}$-forms: $G \cong \mathrm{SL}_1(A)$ for a central simple $\mathbb{Q}$-algebra A of degree 3, where*

   $$\mathrm{SL}_1(A)(R) = \{x \in (A \otimes_{\mathbb{Q}} R)^{\times} : \mathrm{Nrd}(x) = 1\}.$$

   There are two subcases:

      i. *Split case: $A \simeq M_3(\mathbb{Q})$. This yields the commensurability class of $\mathrm{SL}_3(\mathbb{Z})$.*

      ii. *Non-split case: A is a central division $\mathbb{Q}$-algebra of degree 3. In this case a lattice is commensurable to $\mathrm{SL}_1(\mathcal{O})$ for a $\mathbb{Z}$-order $\mathcal{O} \subset A$.*

   (b) *Outer $\mathbb{Q}$-forms: $G \cong \mathrm{SU}(B, \tau)$, where $K/\mathbb{Q}$ is* real quadratic, *$\mathcal{O}_K$ denotes the ring of integers of K, and B is a central simple K-algebra of degree 3, and τ is a* unitary involution *on B (i.e. an involution restricting to the non-trivial automorphism of $K/\mathbb{Q}$). In this case a lattice is commensurable to the group of $\mathbb{Z}$-points of a suitable $\mathbb{Z}$-model of $\mathrm{SU}(B, \tau)$ (equivalently: to $\mathrm{SU}(\mathcal{O}, \tau)$ for a τ-stable $\mathcal{O}_K$-order $\mathcal{O} \subset B$). Moreover, by Lemma 3.4, there are two subcases:*

      i. *Split case: $B \simeq M_3(K)$. Fix an identification $B = M_3(K)$. Then τ is adjoint to a nondegenerate ι-hermitian form on K^3: there exists $H \in \mathrm{GL}_3(K)$ with ${}^{\iota}H^{\mathrm{t}} = H$ such that*

      $$\tau(g) = H^{-1}\, {}^{\iota}g^{\mathrm{t}}\, H \qquad (g \in M_3(K)).$$

Hence G is $\mathbb{Q}$-isomorphic to

$$\mathrm{SU}(H) = \Big\{ g \in \mathrm{SL}_3(K) : {}^{\iota}g^{\mathsf{t}} H g = H \Big\}.$$

In this case, every arithmetic subgroup attached to G is commensurable with

$$\mathrm{SU}(H)(\mathcal{O}_K) := \mathrm{SU}(H) \cap \mathrm{SL}_3(\mathcal{O}_K)$$

for a suitable choice of such an H with entries in $\mathcal{O}_K$. The group $\mathrm{SU}_3(\mathcal{O}_K)$ is a special case, corresponding to $H = I_3$.

ii. Non-split case: B is a central division K-algebra of degree 3 equipped with the unitary involution τ. This yields a non-standard unitary lattice, commensurable to

$$\mathrm{SU}(\mathcal{O}, \tau) = \big\{ u \in \mathcal{O}^{\times} : \tau(u)\, u = 1 \text{ and } \mathrm{Nrd}(u) = 1 \big\}$$

for a τ-stable $\mathcal{O}_K$-order $\mathcal{O} \subset B$.

3. *Commensurability classes of lattices in $\mathrm{SL}_3(\mathbb{R})$ are in bijection with $\mathbb{Q}$-isomorphism classes of simply connected absolutely almost simple $\mathbb{Q}$-groups G with $G(\mathbb{R}) \cong \mathrm{SL}_3(\mathbb{R})$ up to the diagram automorphism of type A_2.*

   In particular, from a lattice $\Gamma < \mathrm{SL}_3(\mathbb{R})$ one can recover:

   (a) whether the ambient $\mathbb{Q}$-form is of inner type or outer type, split or non-split;

   (b) in the inner non-split case, the central simple $\mathbb{Q}$-algebra A of degree 3 such that $G \cong \mathrm{SL}_1(A)$;

   (c) in the outer case, the real quadratic extension $K/\mathbb{Q}$ over which G becomes an inner form and in the non-split case, the central simple K-algebra B of degree 3 with a unitary involution τ.

Sections 6 and 7 prove (1)–(3) while uniqueness is discussed in more detail in Section 7.3. By Theorem 4.2, to classify lattices in $H = \mathrm{SL}_3(\mathbb{R})$ up to commensurability, it suffices to:

1. list the $\mathbb{Q}$-forms G of SL_3 with $G(\mathbb{R}) \cong \mathrm{SL}_3(\mathbb{R})$,
2. use Lemma 2.9 to conclude that each such G yields one commensurability class of lattices,
3. understand the uniqueness of G in terms of the lattice.

The language that organizes this systematically is (non-abelian) Galois cohomology. The topic of the next section is to provide the necessary background on Galois cohomology and explain how it classifies $\mathbb{Q}$-forms of SL_3 in concrete terms. We try to be as self-contained as possible, but for more details and background on Galois cohomology, see [13, 4].

6 Galois cohomology and forms

In this section we explain how Galois cohomology classifies $\mathbb{Q}$-forms of SL_3.

6.1 Non-abelian Galois cohomology: the definition

Let Γ be a profinite group acting continuously on a (not necessarily abelian) topological group A.

Definition 6.1. A *(continuous) 1-cocycle* is a map $z : \Gamma \to A$ such that

$$z_{\sigma\tau} = z_\sigma \cdot \sigma(z_\tau) \qquad (\sigma, \tau \in \Gamma).$$

Two cocycles z, z' are *cohomologous* if there exists $a \in A$ with

$$z'_\sigma = a^{-1} z_\sigma \, \sigma(a) \qquad (\sigma \in \Gamma).$$

The set of cohomology classes is denoted $H^1(\Gamma, A)$; it is a pointed set (basepoint given by the trivial cocycle).

Remark 6.2. For a perfect field k (in particular for number fields) with algebraic closure $\bar{k}$, one writes $H^1(k, A)$ for $H^1(\mathrm{Gal}(\bar{k}/k), A(\bar{k}))$ when A is an algebraic k-group. Background: [13, 4].

6.2 $\mathbb{Q}$-forms of SL_3 via twisting

We are ready to explain how Galois cohomology classifies $\mathbb{Q}$-forms of SL_3.

Theorem 6.3. *Isomorphism classes of $\mathbb{Q}$-forms of SL_3 are in bijection with $H^1(\mathbb{Q}, \mathrm{Aut}(\mathrm{SL}_3))$. More precisely, a cocycle $z \in Z^1(\mathbb{Q}, \mathrm{Aut}(\mathrm{SL}_3))$ defines a twisted form ${}^z\mathrm{SL}_3$, and two cocycles define isomorphic $\mathbb{Q}$-forms if and only if they are cohomologous.*

The preceding theorem is conceptually clean but can feel abstract at first. We now spell out how a cocycle produces an actual $\mathbb{Q}$-group, and how to work with it concretely.

Definition 6.4. Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and write $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Base change gives a $\overline{\mathbb{Q}}$-group $(\mathrm{SL}_3)_{\overline{\mathbb{Q}}}$, and G acts on $(\mathrm{SL}_3)_{\overline{\mathbb{Q}}}$ through its action on $\overline{\mathbb{Q}}$.

Given a 1-cocycle $z\colon G \to \mathrm{Aut}((\mathrm{SL}_3)_{\overline{\mathbb{Q}}})$, $z \in Z^1(\mathbb{Q}, \mathrm{Aut}(\mathrm{SL}_3)) = Z^1(G, \mathrm{Aut}((\mathrm{SL}_3)_{\overline{\mathbb{Q}}}))$, define a new *twisted* action of G on the $\overline{\mathbb{Q}}$-group $(\mathrm{SL}_3)_{\overline{\mathbb{Q}}}$ by

$$\sigma \star_z g \;:=\; z_\sigma\big(\sigma(g)\big) \qquad (\sigma \in G,\ g \in (\mathrm{SL}_3)_{\overline{\mathbb{Q}}}).$$

The cocycle identity $z_{\sigma\tau} = z_\sigma \cdot \sigma(z_\tau)$ is exactly what ensures that $\star$ is an action. The *twisted form* ${}^z\mathrm{SL}_3$ is the unique $\mathbb{Q}$-form of SL_3 whose base change to $\overline{\mathbb{Q}}$ is identified with $(\mathrm{SL}_3)_{\overline{\mathbb{Q}}}$ in such a way that the natural G-action coming from the k-structure corresponds to the twisted action.

In practice one often works over a finite Galois extension $L/\mathbb{Q}$ over which the cocycle is defined. Concretely, choose $L/\mathbb{Q}$ finite Galois and a cocycle $z \in Z^1(\mathrm{Gal}(L/\mathbb{Q}), \mathrm{Aut}(\mathrm{SL}_3)(L))$. Then one may view ${}^z\mathrm{SL}_3$ as a subgroup of SL_3 after base change to L: its $\mathbb{Q}$-points are

$${}^z\mathrm{SL}_3(\mathbb{Q}) \;\cong\; \big\{\, g \in \mathrm{SL}_3(L) \;:\; z_\sigma\big(\sigma(g)\big) = g \text{ for all } \sigma \in \mathrm{Gal}(L/\mathbb{Q}) \,\big\}.$$

More generally, for a $\mathbb{Q}$-algebra R one can describe ${}^z\mathrm{SL}_3(R)$ inside $\mathrm{SL}_3(R \otimes_{\mathbb{Q}} L)$ via the same fixed-point condition. This is a convenient computational model: one works in $\mathrm{SL}_3(L)$ and imposes twisted Galois invariance.

The following proposition shows that cohomologous cocycles yield isomorphic forms, so the construction of ${}^z\mathrm{SL}_3$ depends only on the cohomology class of z.

Proposition 6.5. *Let $z, z' \in Z^1(\mathbb{Q}, \mathrm{Aut}(\mathrm{SL}_3))$. If z and z' are cohomologous, i.e. $z'_\sigma = a^{-1} z_\sigma\, \sigma(a)$ for some $a \in \mathrm{Aut}(\mathrm{SL}_3)(\overline{\mathbb{Q}})$, then the twists ${}^z\mathrm{SL}_3$ and ${}^{z'}\mathrm{SL}_3$ are isomorphic as $\mathbb{Q}$-groups.*

Having this description of $\mathbb{Q}$-forms of SL_3 in terms of Galois cohomology, the classification problem reduces to computing $H^1(\mathbb{Q}, \mathrm{Aut}(\mathrm{SL}_3))$ and understanding the resulting forms, i.e., we need to *find* $\mathbb{Q}$-forms of SL_3, one therefore:

1. determine (or at least describe) the Galois cohomology set $H^1(\mathbb{Q}, \mathrm{Aut}(\mathrm{SL}_3))$;
2. for each class, choose an explicit cocycle representative z;
3. construct the corresponding form ${}^z\mathrm{SL}_3$ by twisting, using Definition 6.4 or the finite-extension recipe above.

In the next subsection, the exact sequence $1 \to \mathrm{PGL}_3 \to \mathrm{Aut}(\mathrm{SL}_3) \to \mathbb{Z}/2\mathbb{Z} \to 1$ organizes the computation into inner vs. outer forms. A detailed proof is standard in the literature; see [13, Ch. III, §1] or [7].

6.3 Computation of $H^1(\mathbb{Q}, \mathrm{Aut}(\mathrm{SL}_3))$: inner vs. outer forms

Over an algebraically closed field, $\mathrm{Aut}(\mathrm{SL}_3)$ fits into

$$1 \longrightarrow \mathrm{PGL}_3 \longrightarrow \mathrm{Aut}(\mathrm{SL}_3) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1,$$

where the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ is induced by the diagram automorphism. References: [13] and standard texts on reductive groups (e.g. [7, 3]).

The induced map $H^1(\mathbb{Q}, \mathrm{Aut}(\mathrm{SL}_3)) \to H^1(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$ splits the discussion into:

- *inner forms* (trivial image in $H^1(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$),
- *outer forms* (non-trivial image), whose minimal field of innerity is a quadratic extension.

Recall the notion of central simple algebras and reduced norm (Definition 3.1), unitary involutions (Definition 3.2), and the associated special unitary group $\mathrm{SU}(B, \tau)$.

Theorem 6.6. *Let G be a simply connected absolutely almost simple algebraic group over $\mathbb{Q}$ of type A_2. Then exactly one of the following holds:*

1. *inner forms: $G \simeq \mathrm{SL}_1(A)$ for a central simple $\mathbb{Q}$-algebra A of degree 3.*
2. *outer forms: $G \simeq \mathrm{SU}(B, \tau)$ for a quadratic extension $K/\mathbb{Q}$, a central simple K-algebra B of degree 3, and a unitary involution τ on B.*

Sketch of proof. Over $\overline{\mathbb{Q}}$ one has $\mathrm{Aut}(\mathrm{SL}_3) \cong \mathrm{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}$, where the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ is the diagram automorphism (for instance $g \mapsto (g^{-1})^t$). Taking Galois cohomology of the exact sequence $1 \to \mathrm{PGL}_3 \to \mathrm{Aut}(\mathrm{SL}_3) \to \mathbb{Z}/2\mathbb{Z} \to 1$ gives a map $H^1(\mathbb{Q}, \mathrm{Aut}(\mathrm{SL}_3)) \to H^1(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$. Its value distinguishes the two possibilities.

If the image is trivial, the class comes from $H^1(\mathbb{Q}, \mathrm{PGL}_3)$, and the latter classifies central simple $\mathbb{Q}$-algebras of degree 3 (up to Brauer equivalence); the associated simply connected inner form is the norm-one group $\mathrm{SL}_1(A)$.

If the image is non-trivial, it corresponds to a quadratic extension $K/\mathbb{Q}$. After base change to K the class becomes inner, hence is given by a degree-3 central simple K-algebra B. Descending back to $\mathbb{Q}$ amounts to specifying the nontrivial Galois action on the center, which is encoded by an involution τ of the second kind on B (i.e. estricting to the nontrivial element of $\mathrm{Gal}(K/\mathbb{Q})$ on $K = Z(B)$). The resulting simply connected $\mathbb{Q}$-group is the special unitary group $\mathrm{SU}(B, \tau)$. □

This is part of the general Tits classification of almost simple groups over fields. Concrete descriptions of the two types appear already in [14]; modern algebraic treatments via central simple algebras and involutions can be found in [10, 5].

6.4 When do we have $G(\mathbb{R}) \cong \mathrm{SL}_3(\mathbb{R})$?

It remains to discuss, which of these have real points $\mathrm{SL}_3(\mathbb{R})$. This is done for inner forms in the following lemma:

Lemma 6.7. *We have* $\mathrm{Br}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$*. In particular, every central simple* $\mathbb{R}$*-algebra of degree* 3 *is split: it is isomorphic to* $M_3(\mathbb{R})$*.*

This is standard; see [4, §4.4].

Proposition 6.8. *If* $A/\mathbb{Q}$ *is central simple of degree* 3*, then* $A \otimes_{\mathbb{Q}} \mathbb{R} \cong M_3(\mathbb{R})$*, hence* $\mathrm{SL}_1(A)(\mathbb{R}) \cong \mathrm{SL}_3(\mathbb{R})$.

Proof. The Brauer class of $A \otimes_{\mathbb{Q}} \mathbb{R}$ has order dividing 3 in $\mathrm{Br}(\mathbb{R})$, hence is trivial by Lemma 6.7. □

For outer forms, the criterion is as follows:

Lemma 6.9. *Let* $K/\mathbb{Q}$ *be a quadratic extension, let* B *be a central simple* K*-algebra of degree* 3*, and let* τ *be a unitary involution on* B*. Put* $G = \mathrm{SU}(B, \tau)$*. Then* $G(\mathbb{R}) \cong \mathrm{SL}_3(\mathbb{R})$ *if and only if* K *is real quadratic. In this case* B *necessarily splits at both real places of* K*.*

Proof. If K is imaginary quadratic then $K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}$, and the restriction of τ to $K \otimes_{\mathbb{Q}} \mathbb{R}$ is complex conjugation. In that case $G(\mathbb{R})$ is (up to inner twist) a real unitary group $\mathrm{SU}(p, q)$ with $p + q = 3$, hence it is not isomorphic to $\mathrm{SL}_3(\mathbb{R})$.

If K is real quadratic then $K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R} \times \mathbb{R}$, and the restriction of τ swaps the two factors. Writing $B \otimes_{\mathbb{Q}} \mathbb{R} \cong B_1 \times B_2$ accordingly, each B_i is a central simple $\mathbb{R}$-algebra of degree 3. It is a standard fact that every central simple $\mathbb{R}$-algebra of degree 3 is split; hence each $B_i \cong M_3(\mathbb{R})$ (equivalently, B splits at both real places of K). The resulting real group is (non-canonically) isomorphic to $\mathrm{SL}_3(\mathbb{R})$. For a systematic treatment of real forms in the unitary-involution language, see [5, Ch. 26] or [10, Ch. 2–3]. □

7 Proof of Theorem 5.1

We now prove Theorem 5.1 by assembling the results accumulated so far. Let $\Gamma < \mathrm{SL}_3(\mathbb{R})$ be a lattice. By Theorem 4.2, Γ is arithmetic: it is commensurable in $\mathrm{SL}_3(\mathbb{R})$ with the image of an arithmetic subgroup of $G(\mathbb{Q})$ for some simply connected absolutely almost simple $\mathbb{Q}$-group G with $G(\mathbb{R}) \cong \mathrm{SL}_3(\mathbb{R})$. Thus the classification problem reduces to describing the possible $\mathbb{Q}$-forms G with these real points and, for each such G, the commensurability class of arithmetic subgroups in $G(\mathbb{R})$ (Lemma 2.9).

Section 6 provides the classification of the relevant $\mathbb{Q}$-forms: by Theorem 6.6 any such G is either an inner form $\mathrm{SL}_1(A)$ or an outer form $\mathrm{SU}(B,\tau)$, and Proposition 6.8 together with Lemma 6.9 ensures that in the outer case the quadratic extension of innerity is real. The following subsections make the corresponding arithmetic lattices explicit in terms of (τ-stable) orders and record the uniform/non-uniform criteria.

7.1 Inner forms and orders

Let $A/\mathbb{Q}$ be central simple of degree 3, and let $\mathcal{O} \subset A$ be an order. Define

$$\mathrm{SL}_1(\mathcal{O}) = \{x \in \mathcal{O}^\times : \mathrm{Nrd}(x) = 1\}.$$

By Theorem 2.17 and Proposition 6.8, $\mathrm{SL}_1(\mathcal{O})$ is a lattice in $\mathrm{SL}_3(\mathbb{R})$.

Lemma 7.1. *If $\mathcal{O}, \mathcal{O}' \subset A$ are $\mathbb{Z}$-orders, then $\mathrm{SL}_1(\mathcal{O})$ and $\mathrm{SL}_1(\mathcal{O}')$ are commensurable.*

Proof. This is a special case of Lemma 2.9. Indeed, $\mathrm{SL}_1(\mathcal{O})$ and $\mathrm{SL}_1(\mathcal{O}')$ are arithmetic subgroups of the fixed $\mathbb{Q}$-group $G = \mathrm{SL}_1(A)$: choosing a faithful $\mathbb{Q}$-representation of G (for instance via left multiplication of A on itself), a $\mathbb{Z}$-order defines a $\mathbb{Z}$-lattice stable under multiplication and hence a $\mathbb{Z}$-structure on G. Thus both groups are arithmetic in $G(\mathbb{Q})$, and Lemma 2.9 implies they are commensurable. □

7.2 Outer forms and τ-stable orders

Let $K/\mathbb{Q}$ be real quadratic with ring of integers $\mathcal{O}_K$, and let (B,τ) be as in Theorem 6.6, so that $\mathrm{SU}(B,\tau)(\mathbb{R}) \cong \mathrm{SL}_3(\mathbb{R})$ by Lemma 6.9. Let $\mathcal{O} \subset B$ be a τ-stable $\mathcal{O}_K$-order; such an order exists by Proposition 3.15 (choose a τ-stable maximal order). Define

$$\mathrm{SU}(\mathcal{O},\tau) = \mathrm{SU}(B,\tau)(\mathbb{Q}) \cap \mathcal{O}^\times,$$

interpreted inside any faithful $\mathbb{Q}$-representation of $\mathrm{SU}(B,\tau)$ (equivalently via an integral model). Then $\mathrm{SU}(\mathcal{O},\tau)$ is an arithmetic lattice in $\mathrm{SL}_3(\mathbb{R})$, see Theorem 2.17.

Lemma 7.2. *Let $K/\mathbb{Q}$ be real quadratic with ring of integers $\mathcal{O}_K$, let (B,τ) be as above, and let $\mathcal{O} \subset B$ be a τ-stable $\mathcal{O}_K$-order. Then, after choosing suitable identifications, the group $\mathrm{SU}(\mathcal{O},\tau)$ can be realized as a subgroup of an ordinary integer matrix group. More precisely:*

1. *split case: There exists a nondegenerate τ-hermitian form with Gram matrix $H \in M_3(\mathcal{O}_K)$ such that $\mathrm{SU}(B,\tau)$ is $\mathbb{Q}$-isomorphic to the special unitary group*

$$\mathrm{SU}(H) = \Big\{ g \in \mathrm{SL}_3(K) : {}^{\iota}g^{\mathrm{t}}\, H\, g = H \Big\},$$

where ι is the non-trivial automorphism of $K/\mathbb{Q}$ acting entrywise on matrices. Under this identification, $\mathrm{SU}(\mathcal{O},\tau)$ *is commensurable with* $\mathrm{SU}(H)(\mathcal{O}_K) = \mathrm{SU}(H) \cap \mathrm{SL}_3(\mathcal{O}_K)$*. Choosing a $\mathbb{Z}$-basis of $\mathcal{O}_K$ yields an embedding* $\mathrm{SL}_3(\mathcal{O}_K) \hookrightarrow \mathrm{SL}_6(\mathbb{Z})$*, hence also an embedding of* $\mathrm{SU}(H)(\mathcal{O}_K)$ *(and thus of* $\mathrm{SU}(\mathcal{O},\tau)$ *up to commensurability) into* $\mathrm{SL}_6(\mathbb{Z})$*.*

2. *non-split case: The left-regular action of $\mathcal{O}$ on itself identifies $\mathcal{O}^\times$ with a subgroup of* $\mathrm{GL}_{18}(\mathbb{Z})$*. Under this embedding,* $\mathrm{SU}(\mathcal{O},\tau) = \mathrm{SU}(B,\tau)(\mathbb{Q}) \cap \mathcal{O}^\times$ *is an arithmetic subgroup of the $\mathbb{Q}$-group* $\mathrm{SU}(B,\tau)$ *realized inside* GL_{18}*.*

Proof sketch with references. In the split case, any unitary involution on $M_3(K)$ is adjoint to a nondegenerate hermitian form on K^3; choosing a suitable $\mathcal{O}_K$-lattice makes the Gram matrix integral, giving the displayed matrix description. The embedding $\mathrm{SL}_3(\mathcal{O}_K) \hookrightarrow \mathrm{SL}_6(\mathbb{Z})$ comes from representing multiplication by elements of $\mathcal{O}_K$ on the free $\mathbb{Z}$-module $\mathcal{O}_K \cong \mathbb{Z}^2$. In the non-split case, $\mathcal{O}$ is a free $\mathbb{Z}$-module of rank 18 and left multiplication gives an injective homomorphism $\mathcal{O}^\times \hookrightarrow \mathrm{GL}_{18}(\mathbb{Z})$. See [5, Ch. I, Ch. II] and [10, Ch. 4–6]. □

Lemma 7.3. *Let $K/\mathbb{Q}$ be real quadratic, let (B,τ) be as above, and let $\mathcal{O},\mathcal{O}' \subset B$ be τ-stable $\mathcal{O}_K$-orders. Then* $\mathrm{SU}(\mathcal{O},\tau)$ *and* $\mathrm{SU}(\mathcal{O}',\tau)$ *are commensurable in* $\mathrm{SU}(B,\tau)(\mathbb{Q})$*.*

Proof. Both $\mathrm{SU}(\mathcal{O},\tau)$ and $\mathrm{SU}(\mathcal{O}',\tau)$ are arithmetic subgroups of the fixed $\mathbb{Q}$-group $G = \mathrm{SU}(B,\tau)$ (each is commensurable with the $\mathbb{Z}$-points of a suitable integral model coming from the chosen order). Therefore they are commensurable by Lemma 2.9. □

7.3 Margulis superrigidity and commensurability

We now explain why the *abstract* commensurability class of a lattice in $\mathrm{SL}_3(\mathbb{R})$ determines the ambient $\mathbb{Q}$-form. The key input is Margulis superrigidity (in its strong rigidity form).

Theorem 7.4 (Margulis)**.** *Let $\Gamma_1, \Gamma_2 < \mathrm{SL}_3(\mathbb{R})$ be lattices and let $\varphi : \Gamma_1' \to \Gamma_2'$ be an isomorphism between finite-index subgroups. Then there exists a (continuous) automorphism $\Phi \in \mathrm{Aut}(\mathrm{SL}_3(\mathbb{R}))$ such that $\Phi|_{\Gamma_1'} = \varphi$. In particular, $\Phi(\Gamma_1)$ and Γ_2 are commensurable in $\mathrm{SL}_3(\mathbb{R})$.*

Remark 7.5. Every automorphism of $\mathrm{SL}_3(\mathbb{R})$ is algebraic. More precisely,

$$\mathrm{Aut}(\mathrm{SL}_3(\mathbb{R})) \cong \mathrm{PGL}_3(\mathbb{R}) \rtimes \mathbb{Z}/2\mathbb{Z},$$

where the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ is induced by the diagram automorphism $g \mapsto (g^{-1})^{\mathsf{t}}$. References for Theorem 7.4 include [6, Ch. IX] and [9, Ch. 16].

To pass from commensurability of subgroups of $\mathrm{SL}_3(\mathbb{R})$ to uniqueness of the ambient $\mathbb{Q}$-form, it is convenient to use commensurators. We refer to Section 6 for the definition and for Margulis's commensurator theorem (Theorem 4.5) in the $\mathrm{SL}_3(\mathbb{R})$ case.

Proposition 7.6. *Let $\Gamma_1, \Gamma_2 < \mathrm{SL}_3(\mathbb{R})$ be lattices that are abstractly commensurable. Let G_1, G_2 be the ambient $\mathbb{Q}$-forms given by Theorem 4.2. Then G_1 and G_2 are $\mathbb{Q}$-isomorphic up to the diagram automorphism of type A_2.*

Proof. Choose finite-index subgroups $\Gamma_1' \le \Gamma_1$ and $\Gamma_2' \le \Gamma_2$ and an isomorphism $\varphi : \Gamma_1' \to \Gamma_2'$. By Theorem 7.4, φ extends to an automorphism $\Phi \in \mathrm{Aut}(\mathrm{SL}_3(\mathbb{R}))$. Thus $\Phi(\Gamma_1') = \Gamma_2'$ and in particular $\Phi(\Gamma_1)$ is commensurable with Γ_2. Commensurable subgroups have the same commensurator, hence

$$\mathrm{Comm}_{\mathrm{SL}_3(\mathbb{R})}(\Phi(\Gamma_1)) = \mathrm{Comm}_{\mathrm{SL}_3(\mathbb{R})}(\Gamma_2).$$

Applying Theorem 4.5 to $\Phi(\Gamma_1)$ and to Γ_2 shows that (up to finite index) these commensurators are the images of $G_1(\mathbb{Q})$ and $G_2(\mathbb{Q})$, respectively. Therefore G_1 and G_2 define the same $\mathbb{Q}$-form of SL_3 up to the diagram automorphism. □

Putting arithmeticity together with (strong) superrigidity, one can thus reconstruct the ambient $\mathbb{Q}$-algebraic group from a lattice $\Gamma < \mathrm{SL}_3(\mathbb{R})$, up to the natural equivalences (diagram automorphism and central isogeny). In particular, the abstract commensurability class of Γ determines the $\mathbb{Q}$-form of the *adjoint* group (equivalently: the $\mathbb{Q}$-isogeny class of the simply connected group), and the classification of A_2-forms then turns this into concrete algebra data.

Theorem 7.7. *Let $\Gamma < \mathrm{SL}_3(\mathbb{R})$ be a lattice and let $G/\mathbb{Q}$ be the simply connected absolutely almost simple $\mathbb{Q}$-group with $G(\mathbb{R}) \cong \mathrm{SL}_3(\mathbb{R})$ such that Γ is commensurable with the image of an arithmetic subgroup of $G(\mathbb{Q})$. Then:*

1. *The abstract commensurability class of Γ determines the $\mathbb{Q}$-isomorphism class of the adjoint $\mathbb{Q}$-group G^{ad}, equivalently: the $\mathbb{Q}$-isogeny class of G, up to the diagram automorphism of type A_2.*

2. *Inner case: If G is an inner form, then $G \cong \mathrm{SL}_1(A)$ for a central simple $\mathbb{Q}$-algebra A of degree 3. The Brauer class $[A] \in \mathrm{Br}(\mathbb{Q})[3]$ (hence in particular the dichotomy split $A \cong M_3(\mathbb{Q})$ versus the non-split (division) case) is determined by G^{ad}. Equivalently, A is determined by G up to $\mathbb{Q}$-algebra isomorphism and passage to the opposite algebra A^{op}.*

3. *Outer case: If G is an outer form, then $G \cong \mathrm{SU}(B, \tau)$ for a real quadratic field $K/\mathbb{Q}$, a central simple K-algebra B of degree 3, and a unitary involution τ on B. The isomorphism class of the triple (K, B, τ) is determined by G^{ad} up to the natural equivalences (isomorphism of $K/\mathbb{Q}$, K-algebra isomorphism of B, and conjugacy of τ inside $B^\times$).*

   In the split outer case $B \cong M_3(K)$, giving τ is equivalent to giving a nondegenerate ι-hermitian form on K^3 up to similarity*: after choosing an identification $B = M_3(K)$ there exists $H \in \mathrm{GL}_3(K)$ with ${}^{\iota}H^{\mathrm{t}} = H$ such that*

   $$\tau(g) = H^{-1}\, {}^{\iota}g^{\mathrm{t}}\, H \qquad (g \in M_3(K)).$$

   What is determined by G is the similarity class of this hermitian form (equivalently: the conjugacy class of the involution τ), not a distinguished Gram matrix H: changing K-basis replaces H by ${}^{\iota}g^{\mathrm{t}}Hg$ with $g \in \mathrm{GL}_3(K)$, and scaling replaces H by cH with $c \in K^\times$, without changing the resulting $\mathbb{Q}$-group.

7.4 Uniform and non-uniform lattices

We now discuss the dichotomy between uniform and non-uniform lattices in the classification from Theorem 5.1. The key point is that non-uniformity is detected by unipotent elements, via Godement's compactness criterion.

Definition 7.8. Let G be a semi-simple algebraic group over $\mathbb{Q}$. We say that G is $\mathbb{Q}$-*anisotropic* if G contains no non-trivial $\mathbb{Q}$-split torus (equivalently: $\mathrm{rank}_{\mathbb{Q}}(G) = 0$). Otherwise, G is called $\mathbb{Q}$-*isotropic.*

Theorem 7.9. *Let $\Gamma < \mathrm{SL}_3(\mathbb{R})$ be a lattice and let G be the ambient $\mathbb{Q}$-group from Theorem 4.2, so that $G(\mathbb{R}) \cong \mathrm{SL}_3(\mathbb{R})$ and Γ is commensurable with the image of an arithmetic subgroup of $G(\mathbb{Q})$. Then the following are equivalent:*

1. $\mathrm{SL}_3(\mathbb{R})/\Gamma$ *is compact;*

2. Γ *contains no non-trivial unipotent element;*

3. G *is* $\mathbb{Q}$*-anisotropic.*

Sketch. We give a complete argument for the implication "unipotents $\Rightarrow$ non-compactness" and then invoke Godement's criterion for the converse.

Let $H = \mathrm{SL}_3(\mathbb{R})$. Assume that H/Γ is compact. We claim that there exists an identity neighborhood $\Omega \subset H$ such that

$$g\Gamma g^{-1} \cap \Omega = \{1\} \qquad \text{for all } g \in H.$$

Indeed, if no such Ω existed, we could find sequences $g_n \in H$ and $\gamma_n \in \Gamma \setminus \{1\}$ with $g_n\gamma_n g_n^{-1} \to 1$. Passing to a subsequence, the points $g_n\Gamma \in H/\Gamma$ converge to some $g\Gamma$. Thus there exist $\delta_n \in \Gamma$ with $g_n\delta_n \to g$. Then

$$(g_n\delta_n)(\delta_n^{-1}\gamma_n\delta_n)(g_n\delta_n)^{-1} = g_n\gamma_n g_n^{-1} \to 1.$$

Conjugating by $(g_n\delta_n)^{-1} \to g^{-1}$ shows $\delta_n^{-1}\gamma_n\delta_n \to 1$. Since Γ is discrete, this forces $\delta_n^{-1}\gamma_n\delta_n = 1$ for $n \gg 0$, a contradiction.

Suppose $u \in \Gamma$ is unipotent and $u \neq 1$. Choose $h \in H$ such that huh^{-1} is upper triangular unipotent (this is possible over $\mathbb{R}$). Let

$$a_t = \mathrm{diag}(t^{-1}, 1, t) \in H \qquad (t > 0).$$

If $huh^{-1} = I_3 + N$ with N strictly upper triangular, then conjugation by a_t scales the (i,j)-entry $(i < j)$ by the factor $a_{t,ii}/a_{t,jj} \in \{t^{-1}, t^{-2}\}$, hence $a_t(huh^{-1})a_t^{-1} \to I_3$ as $t \to \infty$. Therefore

$$(a_t h)u(a_t h)^{-1} \to 1.$$

By Step 1 this is impossible if H/Γ were compact. Therefore H/Γ is non-compact, i.e. Γ is non-uniform.

For arithmetic subgroups, Godement's criterion states that H/Γ is compact if and only if Γ contains no non-trivial unipotent element, equivalently if and only if G is $\mathbb{Q}$-anisotropic. See [10, Ch. 8].

□

Corollary 7.10. *In the classification of Theorem 5.1, the lattices arising from the split cases are non-uniform, while the lattices arising from the non-split cases are uniform.*

Proof. In the non-split inner and outer cases, the ambient algebra (A over $\mathbb{Q}$ or B over K) is a division algebra. Hence it contains no non-zero nilpotent element. If x is unipotent in $A^\times$ (or $B^\times$), then $(x-1)^n = 0$ for some n, so $x = 1$. Thus the corresponding arithmetic lattices contain no non-trivial unipotents and are uniform by Step 3.

In the split inner case $G \cong \mathrm{SL}_3$, the arithmetic lattice is commensurable with $\mathrm{SL}_3(\mathbb{Z})$ and contains non-trivial unipotent elements, so it is non-uniform by Step 2. In the split outer case $B \cong M_3(K)$, the corresponding outer $\mathbb{Q}$-form is $\mathbb{Q}$-isotropic (equivalently: it has a proper $\mathbb{Q}$-parabolic, hence non-trivial unipotent elements in any arithmetic subgroup), so these lattices are non-uniform as well. This gives exactly the dichotomy stated in the theorem. □

8 Examples of lattices

In order to illustrate the preceding classification, we now give explicit examples of lattices in $\mathrm{SL}_3(\mathbb{R})$ corresponding to each of the cases in Theorem 5.1.

8.1 Inner split: $\mathrm{SL}_3(\mathbb{Z})$

The standard lattice $\mathrm{SL}_3(\mathbb{Z})$ corresponds to the split inner form $\mathrm{SL}_1(M_3(\mathbb{Q})) \cong \mathrm{SL}_3$.

8.2 Inner non-split: SL_1 of an order in a division algebra

In order to give a concrete example in the inner non-split case, we need to start with a concrete division algebra of degree 3 over $\mathbb{Q}$.

Consider the cyclic algebra (compare [11, 4]):

$$D = (L/\mathbb{Q}, \sigma, 2), \qquad L = \mathbb{Q}(\alpha),\ \alpha^3 + \alpha^2 - 2\alpha - 1 = 0,$$

with $\sigma(\alpha) = \alpha^2 - 2$, and relations

$$u^3 = 2, \qquad ux = \sigma(x)u \ (x \in L).$$

For this specific cyclic algebra one has $2 \notin N_{L/\mathbb{Q}}(L^\times)$ (local obstruction at $p = 2$), hence D is a central *division* algebra of degree 3 over $\mathbb{Q}$. Let

$$\mathcal{O} = \mathbb{Z}[\alpha] \oplus \mathbb{Z}[\alpha]u \oplus \mathbb{Z}[\alpha]u^2$$

be the explicit order.

To get a concrete embedding into $\mathrm{SL}_3(\mathbb{R})$, write the three real embeddings of L as

$$\tau_1(\alpha) = \alpha_1 = 2\cos\frac{2\pi}{7}, \quad \tau_2(\alpha) = \alpha_2 = 2\cos\frac{4\pi}{7}, \quad \tau_3(\alpha) = \alpha_3 = 2\cos\frac{6\pi}{7},$$

ordered so that $\tau_{i+1} = \tau_i \circ \sigma$ (indices mod 3). Put $\rho = \sqrt[3]{2}$ and

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then a splitting map $\iota : D \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} M_3(\mathbb{R})$ is given on generators by

$$\iota(x) = \mathrm{diag}(\tau_1(x), \tau_2(x), \tau_3(x)) \ (x \in L), \qquad \iota(u) = \rho P = \begin{pmatrix} 0 & \rho & 0 \\ 0 & 0 & \rho \\ \rho & 0 & 0 \end{pmatrix}.$$

One checks $\iota(u)^3 = 2I_3$ and $\iota(u)\iota(x) = \iota(\sigma(x))\iota(u)$. Hence

$$\Gamma := \iota\big(\mathrm{SL}_1(\mathcal{O})\big) \subset \mathrm{SL}_3(\mathbb{R})$$

is an arithmetic lattice (uniform, since D is division).

This is exactly the *inner non-split* case in the classification developed earlier: by Theorem 6.6, the ambient $\mathbb{Q}$-group is an inner form $G = \mathrm{SL}_1(D)$ of type ${}^1A_2$; by Theorem 5.1 this determines one commensurability class of lattices in $\mathrm{SL}_3(\mathbb{R})$; and by Theorem 7.9, lattices in this class are uniform precisely because D is a division algebra (equivalently, non-split over $\mathbb{Q}$).

In coordinates, if $g = x_0 + x_1 u + x_2 u^2 \in D$ with $x_i \in L$, then

$$\iota(g) = \begin{pmatrix} \tau_1(x_0) & \rho\, \tau_1(x_1) & \rho^2\, \tau_1(x_2) \\ \rho^2\, \tau_2(x_2) & \tau_2(x_0) & \rho\, \tau_2(x_1) \\ \rho\, \tau_3(x_1) & \rho^2\, \tau_3(x_2) & \tau_3(x_0) \end{pmatrix},$$

and $\det(\iota(g)) = \mathrm{Nrd}_D(g)$.

8.3 Outer split: an example (the standard form $H = I_3$)

Choose

$$K = \mathbb{Q}(\sqrt{5}), \qquad \mathcal{O}_K = \mathbb{Z}[\omega],\ \omega = \frac{1+\sqrt{5}}{2},$$

that is,

$$\mathcal{O}_K = \{a + b\omega :\ a, b \in \mathbb{Z}\}.$$

with Galois involution $\iota(\sqrt{5}) = -\sqrt{5}$. Set $B = M_3(K)$ and

$$\tau(g) = {}^{\iota}g^{\mathsf{t}}.$$

Then $G = \mathrm{SU}(B, \tau)$ is an outer $\mathbb{Q}$-form of type ${}^2A_2$, split over K. This is the special case of the split outer construction where the associated ι-hermitian form has Gram matrix $H = I_3$. The arithmetic subgroup is

$$\Gamma = \mathrm{SU}_3(\mathcal{O}_K) = \{g \in \mathrm{SL}_3(\mathcal{O}_K) : {}^{\iota}g^{\mathsf{t}} g = I_3\}.$$

Let $v_1, v_2 : K \hookrightarrow \mathbb{R}$ be the two real embeddings,

$$v_1(\sqrt{5}) = +\sqrt{5}, \qquad v_2(\sqrt{5}) = -\sqrt{5}.$$

The real-point identification is concrete:

$$G(\mathbb{R}) = \{(X, Y) \in \mathrm{SL}_3(\mathbb{R}) \times \mathrm{SL}_3(\mathbb{R}) : Y^{\mathsf{t}} X = I_3\} \xrightarrow{\sim} \mathrm{SL}_3(\mathbb{R}), \quad (X, Y) \mapsto X.$$

For $g = (g_{ij}) \in \Gamma$ with $g_{ij} = a_{ij} + b_{ij}\omega$ ($a_{ij}, b_{ij} \in \mathbb{Z}$), the embedding

$$\iota : \Gamma \hookrightarrow \mathrm{SL}_3(\mathbb{R})$$

is

$$\iota(g) = \big(v_1(g_{ij})\big)_{i,j} = \big(a_{ij} + b_{ij}\tfrac{1+\sqrt{5}}{2}\big)_{i,j}.$$

Example element:

$$\varepsilon = \frac{3+\sqrt{5}}{2} \in \mathcal{O}_K^{\times}, \quad N_{K/\mathbb{Q}}(\varepsilon) = 1,$$

so

$$g_\varepsilon = \mathrm{diag}(\varepsilon, \varepsilon^{-1}, 1) \in \Gamma, \quad \iota(g_\varepsilon) = \mathrm{diag}\left(\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}, 1\right).$$

8.4 Outer non-split: unitary lattice from a division algebra over K

Keep $K = \mathbb{Q}(\sqrt{5})$ with non-trivial automorphism $\iota(\sqrt{5}) = -\sqrt{5}$. Since $4^2 - 5 \cdot 1^2 = 11$, the elements

$$\pi := 4 + \sqrt{5}, \qquad \pi' := \iota(\pi) = 4 - \sqrt{5}$$

satisfy $N_{K/\mathbb{Q}}(\pi) = 11$, hence $(11) = (\pi)(\pi')$ in $\mathcal{O}_K$. Define

$$a := \frac{\pi}{\pi'} = \frac{4+\sqrt{5}}{4-\sqrt{5}} \in K^\times, \qquad \text{so that} \qquad \iota(a) = a^{-1}.$$

Let $L_0 = \mathbb{Q}(\alpha)$ where α is a root of $x^3 + x^2 - 2x - 1$. Then $L_0/\mathbb{Q}$ is cyclic of degree 3 with generator σ determined by $\sigma(\alpha) = \alpha^2 - 2$. Put

$$L := K(\alpha) = K \cdot L_0.$$

Then $[L : K] = 3$ (since $[L_0 : \mathbb{Q}] = 3$ and $[K : \mathbb{Q}] = 2$ imply $L_0 \cap K = \mathbb{Q}$), and L/K is cyclic with Galois group generated by the extension of σ acting trivially on K. (Concretely, 11 is unramified in $L_0/\mathbb{Q}$ (the discriminant is 49) and inert because $x^3 + x^2 - 2x - 1$ has no root mod 11. Since 11 splits in $K/\mathbb{Q}$ as $(11) = (\pi)(\pi')$, it follows that for each of the two primes $\mathfrak{p} = (\pi)$ and $\mathfrak{p}' = (\pi')$ of K above 11 the local extension $L_\mathfrak{p}/K_\mathfrak{p}$ is unramified of degree 3.)

Now define the cyclic K-algebra

$$B := (L/K, \sigma, a)$$

with underlying K-vector space $B = L \oplus Lu \oplus Lu^2$ and relations

$$u^3 = a, \qquad ux = \sigma(x)u \ (x \in L).$$

This B is non-split (hence division): at $\mathfrak{p} = (\pi)$ the extension $L_\mathfrak{p}/K_\mathfrak{p}$ is unramified of degree 3 and $v_\mathfrak{p}(a) = 1$. But in an unramified extension E/F of local fields of degree 3 one has $v_F(N_{E/F}(y)) \in 3\mathbb{Z}$ for every $y \in E^\times$. Therefore $a \notin N_{L_\mathfrak{p}/K_\mathfrak{p}}(L_\mathfrak{p}^\times)$, hence $B \otimes_K K_\mathfrak{p}$ is a division algebra. The same holds at $\mathfrak{p}'$ with $v_{\mathfrak{p}'}(a) = -1$.

The element ι extends to an automorphism of L by acting on K and fixing α. Since this extension commutes with σ and $\iota(a) = a^{-1}$, the rules

$$\tau|_L = \iota, \qquad \tau(u) = u^{-1}$$

define a unitary involution on B (in the sense of Definition 3.2). This gives a concrete instance of the outer non-split objects in Theorem 6.6 and therefore contributes to the outer non-split commensurability classes in Theorem 5.1.

Choose a τ-stable $\mathcal{O}_K$-order $\mathcal{O} \subset B$ and define

$$\Gamma := \mathrm{SU}(\mathcal{O}, \tau) = \{x \in \mathcal{O}^\times : \tau(x)x = 1,\ \mathrm{Nrd}(x) = 1\}.$$

At infinity, write the two real embeddings of K as

$$v_1(\sqrt{5}) = +\sqrt{5}, \qquad v_2(\sqrt{5}) = -\sqrt{5}.$$

Let $\tau_1, \tau_2, \tau_3 : L_0 \hookrightarrow \mathbb{R}$ be the three real embeddings, ordered so that $\tau_{i+1} = \tau_i \circ \sigma$. For $j \in \{1, 2\}$ define real embeddings $\tau_{i,j} : L \hookrightarrow \mathbb{R}$ by

$$\tau_{i,j}(k) = v_j(k) \ (k \in K), \qquad \tau_{i,j}(\alpha) = \tau_i(\alpha).$$

We will embed the lattice into $\mathrm{SL}_3(\mathbb{R})$ using only the first real component (so only the maps $\tau_{i,1}$ appear in the final 3×3 matrix formula), but the maps $\tau_{i,2}$ are needed to write the full splitting isomorphism $B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_3(\mathbb{R}) \times M_3(\mathbb{R})$ and to describe how the unitary involution τ swaps the two real factors. Put $\rho_j := \sqrt[3]{v_j(a)} \in \mathbb{R}_{>0}$ and let $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Then an explicit splitting isomorphism

$$\iota_\infty : B \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} M_3(\mathbb{R}) \times M_3(\mathbb{R})$$

is given on generators by

$$\iota_\infty(x) = \Big(\mathrm{diag}(\tau_{1,1}(x), \tau_{2,1}(x), \tau_{3,1}(x)),\ \mathrm{diag}(\tau_{1,2}(x), \tau_{2,2}(x), \tau_{3,2}(x))\Big)\ (x \in L),$$

$$\iota_\infty(u) = (\rho_1 P, \rho_2 P).$$

Under this identification, τ becomes $(X, Y) \mapsto (Y^{\mathsf{t}}, X^{\mathsf{t}})$. Hence

$$\mathrm{SU}(B, \tau)(\mathbb{R}) = \{(X, Y) : Y^{\mathsf{t}} X = I_3,\ \det(X)\det(Y) = 1\} \xrightarrow{\sim} \mathrm{SL}_3(\mathbb{R}),\ (X, Y) \mapsto X.$$

So the concrete embedding

$$\iota : \Gamma \hookrightarrow \mathrm{SL}_3(\mathbb{R})$$

is: send $x \in \Gamma$ to the first 3×3 real matrix component X of $\iota_\infty(x)$.

Concretely, for $g = x_0 + x_1 u + x_2 u^2 \in B$ with $x_i \in L$, the first component of $\iota_\infty(g)$ is

$$\iota(g) = \begin{pmatrix} \tau_{1,1}(x_0) & \rho_1\, \tau_{1,1}(x_1) & \rho_1^2\, \tau_{1,1}(x_2) \\ \rho_1^2\, \tau_{2,1}(x_2) & \tau_{2,1}(x_0) & \rho_1\, \tau_{2,1}(x_1) \\ \rho_1\, \tau_{3,1}(x_1) & \rho_1^2\, \tau_{3,1}(x_2) & \tau_{3,1}(x_0) \end{pmatrix}.$$

This gives an explicit (typically uniform) outer non-split lattice in $\mathrm{SL}_3(\mathbb{R})$.

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