Classification of minimal non-spherical Coxeter diagrams

Abstract. We recall the classification of Coxeter diagrams for which the Coxeter group is infinite but every proper induced subdiagram has finite Coxeter group.

1. Coxeter diagrams

Finite Coxeter systems are classified by their Coxeter diagrams. We study the minimal diagrams outside the finite case: Coxeter diagrams whose group is infinite and for which every proper induced subdiagram has finite Coxeter group. We call such a diagram *minimal non-spherical*, see below for details.

1.1. Definitions and the bilinear form.

Definition 1.1. A Coxeter system is a pair (W, S), where W is a group with generating set S, such that

$$W = \langle S \mid (st)^{m_{st}} = 1 \text{ whenever } m_{st} < \infty \rangle ,$$

where $m_{ss} = 1$ and $m_{st} = m_{ts} \in \{2, 3, \ldots, \infty\}$ for $s \neq t$ [**1**, Theorem 2.49 and Definition 2.50].

Definition 1.2. The Coxeter diagram of (W, S) is the undirected graph with vertex set S in which distinct vertices s, t are joined when $m_{st} \geq 3$. The edge is unlabeled when $m_{st} = 3$ and is labeled by m_{st} when $m_{st} \geq 4$, including $m_{st} = \infty$. The rank of the Coxeter diagram is $|S|$, see [**1**, §1.5.6].

For $J \subseteq S$, let $W_J = \langle J \rangle$. This is the *standard parabolic subgroup* generated by J [**1**, Definition 2.12]. The following result is standard, see [**1**, Exercise 2.51 and the discussion following Proposition 3.16].

Proposition 1.3. *The pair (W_J, J) is a Coxeter system whose Coxeter matrix is $(m_{st})_{s,t \in J}$. Its Coxeter diagram is the induced subdiagram on J.*

The cosine or Tits matrix of a Coxeter diagram defined to be

$$B = (b_{st})_{s,t \in S}, \qquad b_{ss} = 1, \qquad b_{st} = -\cos\left(\frac{\pi}{m_{st}}\right),$$

with $\cos(\pi/\infty) = \cos 0 = 1$.

A Coxeter diagram is called *spherical* if W is finite. A standard result is that this is equivalent to B being positive definite [**1**, Corollary 2.68 and Definition 2.70]. It is called *minimal non-spherical* if it is not spherical but every proper induced subdiagram is spherical. Equivalently, every proper standard parabolic subgroup is finite.

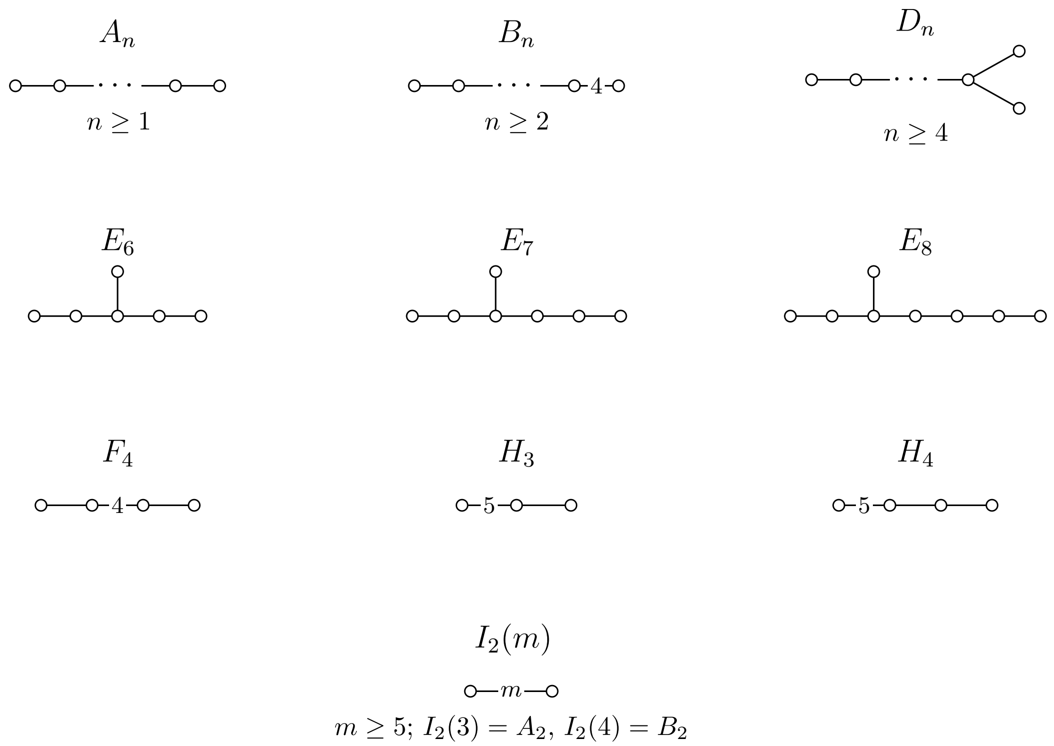
A connected Coxeter diagram is said to be *affine* if B is positive semidefinite and degenerate. In this case B has corank one [**1**, Proposition 10.44]. A connected Coxeter diagram of rank n is called *compact hyperbolic* if B has signature $(n-1, 1)$

and every proper induced subdiagram is spherical. Such a diagram is also called a *Lannér diagram*.

Note that a minimal non-spherical Coxeter diagram is connected. Indeed, if all connected components were spherical, then W would be finite; if one component were non-spherical, that component would be a proper non-spherical induced subdiagram.

1.2. Connected spherical Coxeter diagrams. Every spherical Coxeter diagram is a disjoint union of the connected types below, where unlabeled edges have label 3. See [**1**, Table 1.1] and also Humphreys' Figure 1 in §2.4 [**3**].

In a family diagram, $\cdots$ denotes the chain or cycle length determined by the stated rank; labelled edges and attached branches are retained. Thus A_1 is one vertex, $\widetilde{A}_2$ is a triangle, and $\widetilde{D}_4$ is a four-armed star.



A Coxeter diagram is called *crystallographic* if $m_{st} \in \{2, 3, 4, 6, \infty\}$ for all distinct s, t; equivalently, every finite edge label is 3, 4, or 6. Thus the non-crystallographic spherical types are H_3, H_4, and $I_2(m)$ with $m \geq 3$ and $m \notin \{3, 4, 6\}$. In crystallographic notation $I_2(6)$ is G_2; as a Coxeter type it is already contained in the dihedral family. For all this, see [**2**, Ch. VI, §4, no. 1, Theorem 3, and Planches I–IX].

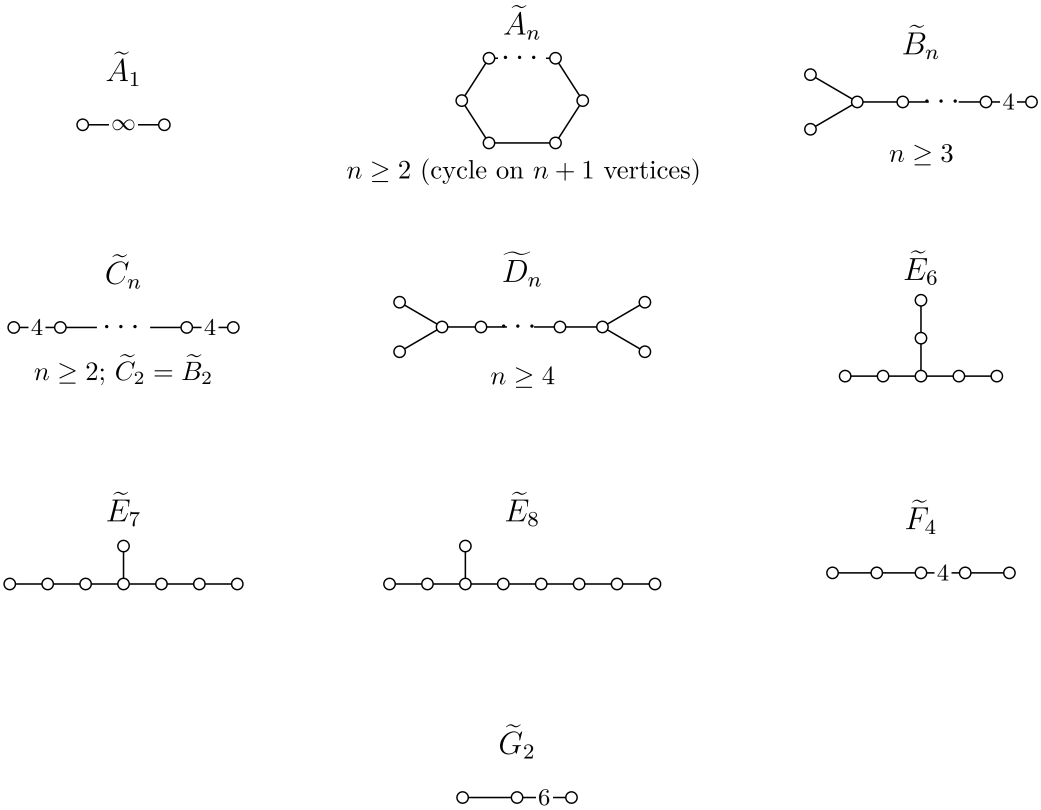
1.3. The minimal non-spherical dichotomy.

Proposition 1.4. *Let B be the cosine matrix of a connected minimal non-spherical Coxeter diagram of rank n. Then exactly one of the following occurs:*

(i) *B is positive semidefinite with* $\operatorname{corank} B = 1$*; the diagram is irreducible affine;*
(ii) *B has signature $(n-1, 1)$; the diagram is compact hyperbolic, equivalently a Lannér diagram.*

Proof. Every $(n-1) \times (n-1)$ principal submatrix is positive definite. Write the eigenvalues of B as $\lambda_1 \leq \cdots \leq \lambda_n$, and those of any such principal submatrix as $\mu_1 \leq \cdots \leq \mu_{n-1}$. Cauchy interlacing gives $\lambda_1 \leq \mu_1 \leq \lambda_2$, so $\lambda_2 > 0$. Thus at most one eigenvalue is nonpositive. Since B is not positive definite, either $\lambda_1 = 0$, giving corank one, or $\lambda_1 < 0$, giving signature $(n-1, 1)$. The first case is the irreducible affine case. In the second case, [**1**, Theorem 10.55] realizes W as a hyperbolic reflection group with a simplicial chamber. Every proper standard parabolic subgroup is finite, so the chamber has no ideal vertices and is compact [**1**, Remark 3.29]. □

1.4. Irreducible affine Coxeter diagrams. The irreducible affine Coxeter diagrams are classified in [**1**, Theorem 10.30 and Remark 10.33]. The drawings follow Humphreys' Figure 2 in §2.5 [**3**]; the coincidence $\widetilde{B}_2 = \widetilde{C}_2$ is shown once.

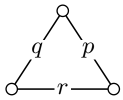


Every irreducible affine Coxeter diagram is crystallographic; equivalently, every irreducible affine Coxeter group is an affine Weyl group. Hence no affine Coxeter diagram contains H_3, H_4, or $I_2(m)$ with $m \geq 3$ and $m \notin \{3, 4, 6\}$ as an induced subdiagram [**1**, Theorem 10.30 and Remark 10.33]; see also [**2**, Ch. VI, §2, no. 1 and §4, no. 3] for affine Weyl groups and completed Coxeter graphs.

1.5. Compact-hyperbolic Coxeter diagrams: Lannér diagrams. A Lannér diagram of rank n is the Coxeter diagram of a compact hyperbolic simplex in $\mathbb{H}^{n-1}$ [**1**, §10.3], [**4, 3**]. There is no rank-two Lannér diagram: a finite edge label gives a spherical dihedral group, whereas the label ∞ gives the affine diagram $\widetilde{A}_1$.

Rank three. There is an infinite family. Up to permutation of the vertices, it is

$$\mathcal{L}(p,q,r), \qquad 2 \le p \le q \le r < \infty, \qquad \frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1.$$

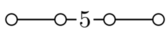


An edge labeled 2 is omitted. Thus paths are included by taking $p = 2$. Equality $1/p + 1/q + 1/r = 1$ is affine; its integral solutions are $(3,3,3)$, $(2,4,4)$, and $(2,3,6)$, corresponding to $\widetilde{A}_2$, $\widetilde{C}_2$, and $\widetilde{G}_2$. The case $1/p + 1/q + 1/r > 1$ is spherical.

In rank at least four no edge has label ∞, and every finite edge label is at most 5. Indeed, an ∞-edge already forms a proper non-spherical rank-two subdiagram. If a finite edge has label $m \ge 6$, connectedness supplies a third vertex adjacent to one of its endpoints; the induced rank-three path or triangle is not spherical, again a contradiction. Hence, in each fixed rank, only finitely many labelled diagrams need be considered.

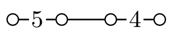
Rank four: nine diagrams. We label the displayed diagrams $L_{4,i}$ and $L_{5,i}$.

$L_{4,1}$



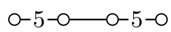
$(3,5,3)$

$L_{4,2}$



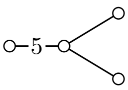
$(5,3,4)$

$L_{4,3}$



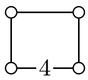
$(5,3,5)$

$L_{4,4}$



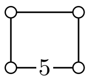
one 5-arm

$L_{4,5}$



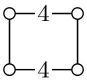
$(4,3,3,3)$

$L_{4,6}$



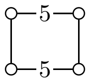
$(5,3,3,3)$

$L_{4,7}$



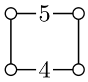
$(4,3,4,3)$

$L_{4,8}$



$(5,3,5,3)$

$L_{4,9}$



$(4,3,5,3)$

Rank five: five diagrams.

$L_{5,1}$

$(5,3,3,3)$

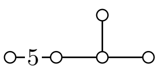
$L_{5,2}$

$(5,3,3,4)$

$L_{5,3}$

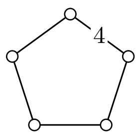
$(5,3,3,5)$

$L_{5,4}$



one 5-arm

$L_{5,5}$



one edge labelled 4

Rank bound. Let Γ be minimal non-spherical with $n \geq 6$ vertices. Since spherical diagrams are forests, Γ is a tree or a chordless cycle. A cycle whose edge labels are all 3 is $\widetilde{A}_{n-1}$; otherwise, deleting a suitable vertex leaves a rank-$(n-1)$ path with an interior edge label at least 4, which is not spherical. For a tree, inspection of the spherical list leaves, beyond the spherical cases, only $\widetilde{B}$, $\widetilde{C}$, $\widetilde{D}$, and $\widetilde{E}$. Thus Γ is affine and cannot be Lannér.

Hence compact hyperbolic Coxeter simplices occur only in dimensions 2, 3, and 4. In dimension one, a compact interval has diagram $\widetilde{A}_1$. For the rank-four and rank-five classifications, see [**3**, §6.9] and [**4**].

2. Sources and finite verification

The basic facts about Coxeter systems, standard parabolic subgroups, induced subdiagrams, and spherical and affine types are taken from Abramenko–Brown [**1**, Chapters 2, 3, and 10]. The connected spherical diagrams are listed in [**1**, Table 1.1] and in Humphreys' Figure 1 in §2.4 [**3**]. The irreducible affine diagrams are given in [**1**, Theorem 10.30 and Remark 10.33] and in Humphreys' Figure 2 in §2.5. For the crystallographic cases, Bourbaki constructs the affine Weyl groups and completed Coxeter graphs in [**2**, Ch. VI, §2, no. 1; §4, nos. 1 and 3; Planches I–IX].

The rank-three Lannér diagrams are the hyperbolic triangles described above. The rank-four and rank-five lists are given in Humphreys' §§6.8–6.9, Figure 2, and in Lannér's classification [**4**].

Because older hyperbolic tables contain misprints [**3**, §6.9], the displayed diagrams are also checked by exact enumeration. The script `verify_classification.py` performs this check. In ranks four and five, the edge-label bound reduces the search to labels $3, 4, 5$. The script enumerates every connected unlabelled graph, forms its cosine matrix over $\mathbb{Q}(\sqrt{2}, \sqrt{5})$, checks the relevant principal determinants exactly, and quotients by graph isomorphism. It obtains exactly nine rank-four and five rank-five Lannér diagrams. It also reconstructs the Coxeter labels of all fourteen displayed diagrams in this document and checks them against the enumeration.

References

[1] P. Abramenko and K. S. Brown, *Buildings: Theory and Applications*, Graduate Texts in Mathematics 248, Springer, New York, 2008; doi:10.1007/978-0-387-78835-7.

[2] N. Bourbaki, *Groupes et algèbres de Lie, Chapitres IV–VI*, Hermann, Paris, 1968; English translation, Springer, 2002. See Chapter VI, §2, no. 1; §4, nos. 1 and 3; and Planches I–IX.

[3] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics 29, Cambridge University Press, 1990. See §§2.4–2.7, 4.7, and 6.8–6.9.

[4] F. Lannér, "On complexes with transitive groups of automorphisms," *Communications du Séminaire Mathématique de l'Université de Lund* **11** (1950), 1–71.

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