

A conditional construction of a nonhyperlinear group and the centralizer problem

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ABSTRACT. We show that a positive answer to the centralizer problem for tracial matrix ultraproducts implies the existence of a finitely generated nonhyperlinear group.

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1. Introduction

In his 1976 classification paper, Connes suggested that every separable II_1 factor should embed into an ultrapower R^ω of the hyperfinite II_1 factor [7]. Kirchberg subsequently placed this question in a broad circle of equivalences involving tensor products and the QWEP problem for C^* -algebras [14]. Work relating the embedding problem to Tsirelson's problem connected it with finite-dimensional quantum correlations [13]. The theorem $\text{MIP}^* = \text{RE}$ of Ji, Natarajan, Vidick, Wright, and Yuen ultimately disproved the Connes embedding problem [12].

The corresponding group-theoretic problem remains open, while a recent breakthrough by OpenAI has produced the first example of a nonsofic group [19], see also [17]. Rădulescu introduced hyperlinear groups [21]: a countable group is hyperlinear if it embeds into the unitary group of a tracial ultraproduct of matrix algebras. Elek and Szabó proved that every sofic group is hyperlinear [10]. Although the general Connes embedding problem has a negative answer, no nonhyperlinear group is known.

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The study of sofic groups is in a sense more approachable, since it allows for input from finite combinatorics. Kun proved that every sofic approximation of a Kazhdan group is, up to a negligible modification, a disjoint union of expanders [15]. Building on this decomposition, Kun and the author showed that the sufficiently accurate almost automorphisms of such an expander sofic approximation form a group in a natural way [16]. Alekseev and the author subsequently proved that, after passing to an essentially equivalent finite model, the centralizer of a sofic embedding of a Kazhdan group in a universal sofic group is itself an internal metric ultraproduct of finite permutation groups [2]. Motivated by these results, they formulated the following matrix analogue [2, Open Problem 6.2(a)], which we call the centralizer problem.

THE CENTRALIZER PROBLEM.

Let G be a Kazhdan group and let $\pi: G \rightarrow U(\prod_{n \rightarrow \mathcal{U}} M_{d_n}(\mathbb{C}))$ be a homomorphism. Do there exist finite-dimensional $*$ -subalgebras $A_n \subseteq M_{d_n}(\mathbb{C})$ such that

$$\pi(G)' \cap \prod_{n \rightarrow \mathcal{U}} M_{d_n}(\mathbb{C}) = \prod_{n \rightarrow \mathcal{U}} A_n?$$

REMARK 1.1. The formulation in [2, Open Problem 6.2(a)] allows first replacing d_n by m_n , where $m_n/d_n \rightarrow_{\mathcal{U}} 1$. This flexibility is in fact unnecessary for the internality assertion above, by Lemma 2.3 below. This contrasts with the situation for sofic approximations and with the correction of approximate representations in normalized Hilbert–Schmidt norm, for which a change of dimension can be essential; see Becker and Lubotzky [4, Appendix A].

The purpose of this paper is to show that a positive answer to this problem implies the existence of a finitely generated nonhyperlinear group. The construction announced by OpenAI [19] inspired Kun and the author to isolate the mechanism used in the proof of [19, Proposition 2.3] in [17]. For a subgroup $H < G$, put $P_H = \{t \in G : tHt^{-1} \leq H\}$, the so-called *compression semigroup* of H in G . Following [17], we call H *infranormal* in G if P_H generates G . They proved that if $H < G$ are Kazhdan, H is infranormal and nonnormal, then both $G *_H (H \times \mathbb{Z}/2\mathbb{Z})$ and the group double $G *_H G$ are nonsofic [17]. The key representation-theoretic statement is that the centralizer of the image of H in a universal sofic group is normalized by the image of G ; see [17, Theorem 4.1].

We prove a matrix counterpart to this result, conditional on a positive answer to the centralizer problem.

THEOREM 1.2. Assume that the centralizer problem has a positive answer. Let $H < G$ be Kazhdan groups and suppose that H is infranormal in G . If $\pi: G \rightarrow U(\mathcal{M})$ is a homomorphism into a tracial matrix ultraproduct, then

$$\pi(g)(\pi(H)' \cap \mathcal{M})\pi(g)^{-1} = \pi(H)' \cap \mathcal{M}$$

for every $g \in G$.

The proof proceeds in two steps: we first prepare exact inclusions at the finite-dimensional level, and then establish a phenomenon which we call *no-drift*. Put $A = \pi(H)' \cap \mathcal{M}$ and $D = \pi(G)' \cap \mathcal{M}$. Choose $t_1, \dots, t_m \in P_H$ such that $G = \langle H, t_1, \dots, t_m \rangle$, and set $u_\ell = \pi(t_\ell)$. Since $t_\ell H t_\ell^{-1} \leq H$, one immediately has $u_\ell^* A u_\ell \subseteq A$. The centralizer problem gives aligned internal models $D_n \subseteq A_n$, with

representatives $u_{\ell,n} \in D'_n$. At finite level, the preceding inclusion becomes the one-sided near inclusion $A_n^- = u_{\ell,n}^* A_n u_{\ell,n} \subset_{2,\varepsilon_n} A_n^+ = A_n$. The relative correction theorem replaces it by an exact inclusion $\hat{A}_n^- \subseteq \hat{A}_n^+$, with compatible identifications of D_n on asymptotically full corners.

Assign to a block $M_{r_i}(\mathbb{C}) \otimes 1_{s_i}$ the scale r_i/s_i . For an exact inclusion these scales are ordered, and a conditional-expectation estimate bounds the reverse near-inclusion error in terms of the scale quotient. A D_n -conditional median gives a bounded transform with conditional expectation $1/2$, and the common copy of D_n allows the same median on the source and target sides. The scale order passes to the ultraproduct, where equality of traces makes this transform invariant under every u_ℓ . It therefore belongs to D and equals $1/2$. The resulting concentration makes the conditional-expectation bound tend to zero, giving the reverse near inclusion and hence $u_\ell^* A u_\ell = A$.

The application is immediate once Theorem 1.2 is available.

THEOREM 1.3. *Assume that the centralizer problem has a positive answer. If $H < G$ is infranormal and not normal, and both groups are Kazhdan, then $G *_H G$ is not hyperlinear.*

Moreover, let q be a prime power and let $r, d \geq 3$. Put $R_+ = \mathbb{F}_q[x_1, \dots, x_d]$ and $R = \mathbb{F}_q[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$, and set $H = \text{EL}_r(R_+)$ and $G = \text{EL}_r(R) \rtimes \text{SL}_d(\mathbb{Z})$, where $\text{SL}_d(\mathbb{Z})$ acts by monomial substitutions. Then $H < G$ satisfies these assumptions.

The pair in Theorem 1.3 is the explicit residually finite Kazhdan pair constructed in [17, Theorem E].

This paper is organized as follows. Section 2 collects the necessary facts about inclusions of internal finite-dimensional algebras. Section 3 proves a relative correction theorem for one-sided near inclusions. Section 4 proves the no-drift theorem and explains why the common internal subalgebra is essential. Section 5 applies the theorem to normalization of Kazhdan centralizers and then to the conditional nonhyperlinearity of a group double.

2. Inclusions of internal algebras

Fix a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. All matrix algebras carry their normalized traces. We write $\mathcal{M} = \prod_{n \rightarrow \mathcal{U}} (M_{d_n}(\mathbb{C}), \text{tr}_n)$ for the tracial ultraproduct and $\|x\|_2 = \text{tr}_n(x^*x)^{1/2}$ at the finite level. If $A_n \subseteq M_{d_n}(\mathbb{C})$ are unital $*$ -subalgebras, then $\prod_{n \rightarrow \mathcal{U}} A_n$ denotes their tracial ultraproduct inside $\mathcal{M}$. These are the usual metric ultraproducts of finite tracial von Neumann algebras; for general background on von Neumann algebra ultraproducts, see [1].

DEFINITION 2.1. A countable group Λ is *hyperlinear* if there are a nonprincipal ultrafilter $\mathcal{U}$, integers d_n , and an injective homomorphism $\Lambda \rightarrow U(\prod_{n \rightarrow \mathcal{U}} M_{d_n}(\mathbb{C}))$.

For finite-dimensional subalgebras $B, A \subseteq M_d(\mathbb{C})$, put

$$B \subset_{2,\varepsilon} A \iff \sup_{b \in B, \|b\| \leq 1} \text{dist}_2(b, A) \leq \varepsilon.$$

Let E_A and E_B be the trace-preserving conditional expectations, and put $\|T\|_{\infty,2} = \sup_{\|x\| \leq 1} \|T(x)\|_2$ for a linear map T . Then

$$(2.1) \quad B \subset_{2,\varepsilon} A \iff \|(1 - E_A)E_B\|_{\infty,2} \leq \varepsilon.$$

Indeed, E_A is the orthogonal projection onto A in L^2 , and the image under E_B of the ambient operator-norm unit ball is precisely the operator-norm unit ball of B . This is the classical notion of ε -containment in 2-norm, introduced by Murray and von Neumann [18] and developed by Christensen [6]. The conditional-expectation formulation in (2.1) is used systematically by Popa, Sinclair, and Smith [20].

The following observation will be used repeatedly.

LEMMA 2.2. *If $A_n, B_n \subseteq M_{d_n}(\mathbb{C})$ and $\prod_{n \rightarrow \mathcal{U}} B_n \subseteq \prod_{n \rightarrow \mathcal{U}} A_n$, then there are $\varepsilon_n \rightarrow_{\mathcal{U}} 0$ such that $B_n \subset_{2, \varepsilon_n} A_n$.*

PROOF. Otherwise, on a set belonging to $\mathcal{U}$, one can choose contractions $b_n \in B_n$ with $\text{dist}_2(b_n, A_n)$ bounded below. The element $(b_n)_{n \rightarrow \mathcal{U}}$ belongs to $\prod_{\mathcal{U}} B_n$ but not to $\prod_{\mathcal{U}} A_n$, a contradiction. $\square$

Replacing d_n by m_n with $m_n/d_n \rightarrow_{\mathcal{U}} 1$, adding or deleting a corner of normalized trace tending to zero, and conjugating by partial isometries on the remaining corner do not change the tracial ultraproduct. We refer to such changes as *stable negligible modifications*. They will always be accompanied by estimates showing that the modified and original unit balls have 2-Hausdorff distance tending to zero.

Once such a modification has been made, we identify the old and new ultraproducts and continue to use the symbols d_n , tr_n , and $\|\cdot\|_2$. Thus harmless changes of normalized trace caused by a dimension ratio tending to one will not be recorded repeatedly.

LEMMA 2.3. *Let $m_n/d_n \rightarrow_{\mathcal{U}} 1$. For every sequence of unital $*$ -subalgebras $A_n \subseteq M_{m_n}(\mathbb{C})$ there are unital $*$ -subalgebras $B_n \subseteq M_{d_n}(\mathbb{C})$ such that, under the canonical identification of the ambient tracial ultraproducts, $\prod_{n \rightarrow \mathcal{U}} A_n = \prod_{n \rightarrow \mathcal{U}} B_n$. Consequently, internality of a subalgebra is invariant under stable negligible changes of matrix dimension.*

PROOF. If the dimensions remain bounded along $\mathcal{U}$, the assumption implies $m_n = d_n$ on a set belonging to $\mathcal{U}$, and there is nothing to prove. We may therefore assume that the dimensions tend to infinity along $\mathcal{U}$. At indices where $m_n \leq d_n$, take $B_n = A_n \oplus \mathbb{C}1_{d_n - m_n}$. The added scalar corner has normalized trace tending to zero.

It remains to treat indices where $m_n > d_n$. Suppress the index n , put $k = m - d$, and write, after unitary conjugacy,

$$A = \bigoplus_{i=1}^t (M_{r_i}(\mathbb{C}) \otimes 1_{s_i}) \subseteq M_m(\mathbb{C}), \quad w_i = r_i s_i.$$

Choose j such that $S := \sum_{i < j} w_i \leq k < S + w_j$, and put $q = k - S$, $h = \min\{r_j, s_j\}$, and $q' = h \lceil q/h \rceil$. Then $0 \leq e := q' - q < h \leq \sqrt{w_j} \leq \sqrt{m}$. Delete the first $j - 1$ summands. In the j -th summand delete q' dimensions by replacing r_j with $r_j - q'/s_j$ if $s_j \leq r_j$, and by replacing s_j with $s_j - q'/r_j$ if $r_j < s_j$. Let p be the support projection of the retained algebra and put $C = pAp$, regarded as an algebra on $p\mathbb{C}^m$. Then

$$\text{rank}(p) = m - S - q' = d - e, \quad \text{rank}(1 - p) = k + e = o_{\mathcal{U}}(m).$$

Write D_1 for the operator-norm unit ball of an algebra D and $d_{H,2}^{(m)}$ for Hausdorff distance in the normalized 2-norm of $M_m(\mathbb{C})$. Compression sends A_1 into C_1 up

to an error supported on $1 - p$, while every element of C_1 has a norm-preserving block extension to A_1 . Hence

$$d_{H,2}^{(m)}(A_1, C_1) \leq 2\sqrt{\frac{k+e}{m}} = o_{\mathcal{U}}(1).$$

Let $Q \in M_m(\mathbb{C})$ be the standard projection of rank d and choose $R \leq Q$ of rank $d - e$. Since both p and R have codimension $k + e = o_{\mathcal{U}}(m)$, there is a unitary $u \in U(m)$ such that $upu^* = R$ and $\|u - 1\|_{2,m} = o_{\mathcal{U}}(1)$. Set

$$\tilde{B} = uCu^* \oplus \mathbb{C}(Q - R) \subseteq QM_m(\mathbb{C})Q \cong M_d(\mathbb{C}),$$

and let $B \subseteq M_d(\mathbb{C})$ be the corresponding algebra. The extra scalar corner has rank $e = o_{\mathcal{U}}(m)$, so the unit balls of A and B have normalized 2-Hausdorff distance tending to zero under the standard corner identification. Equality of the two ultraproduct algebras follows. Interchanging d_n and m_n gives the final assertion. $\square$

3. Correcting near inclusions

There is an extensive perturbation theory for finite von Neumann subalgebras. Besides the foundational work of Christensen [6], the closest general results to what follows are due to Popa, Sinclair, and Smith [20]. Under the symmetric hypothesis $\|E_A - E_B\|_{\infty,2} \leq \delta$, they obtain, for small δ , large cutdowns of A and B that are spatially isomorphic through a partial isometry close to the identity [20, Theorem 5.2]. They also prove that if $A \subseteq B$ is already an exact inclusion and $B \subset_{2,\delta} A$, with $\delta < 23^{-1/2}$, then A and B agree on a corner cut out by a projection $p \in Z(A' \cap B)$ satisfying $\text{tr}(p) \geq 1 - 23\delta^2$ [20, Theorem 3.5]. On the other hand, their one-sided example [20, Proposition 5.5 and the preceding discussion] shows that, for general finite von Neumann algebras, $B \subset_{2,\delta} A$ need not yield even a nonzero spatially embedded corner of B in A .

For a fixed finite-dimensional algebra, a related correction of approximate matrix units in uniform 2-norm is given by Carrión, Castillejos, Evington, Gabe, Schafhauser, Tikuisis, and White [5, Proposition 8.8]. Its tolerance depends on the chosen finite-dimensional source. It therefore does not give a uniform statement for a sequence B_n whose number and sizes of matrix blocks may grow.

What is needed here is consequently a specifically finite-dimensional and stable statement: the near inclusion is only one-sided, the algebras may vary without a complexity bound, a corner of asymptotically negligible dimension may be changed, and a common algebra D_n must be carried to the same corrected copy. These additional features are the content of Proposition 3.1 below. We include the proof for completeness.

Stable correction of approximate representations in normalized 2-norm was developed by Gowers and Hatami for finite groups [11] and by de Chiffre, Ozawa, and the author for amenable groups [9]. Compact-group and unitary-group forms appear in [8, 3]. The relative finite-dimensional statement needed here follows from a Stinespring dilation and a spectral cutoff.

PROPOSITION 3.1. *Let $A_n, B_n \subseteq M_{d_n}(\mathbb{C})$ be unital finite-dimensional algebras and suppose $B_n \subset_{2,\varepsilon_n} A_n$, where $\varepsilon_n \rightarrow_{\mathcal{U}} 0$. After stable negligible modifications there are exact inclusions $\hat{B}_n \subseteq \hat{A}_n$ whose two ultraproducts agree with those of B_n*

and A_n , respectively. If $D_n \subseteq A_n \cap B_n$, the correction can be chosen so that the two identifications carry D_n to the same subalgebra on asymptotically full corners.

PROOF. It suffices to work on the $\mathcal{U}$ -large set where $\varepsilon_n < 1/2$. We first suppress n and use the unnormalized Hilbert–Schmidt norm. The map

$$\Phi : B \otimes A' \longrightarrow M_d(\mathbb{C}), \quad \Phi(b \otimes a) = E_A(b)a,$$

is unital and completely positive: it is the composition of $E_A|_B \otimes \text{id}$ with the multiplication $*$ -homomorphism $A \otimes A' \rightarrow M_d(\mathbb{C})$. Take a finite-dimensional Stinespring dilation [22] $\Phi(x) = V^* \rho(x) V$, where $V : \mathbb{C}^d \rightarrow \mathcal{K}$ is an isometry, and write $\rho_B, \rho_{A'}$ for the commuting factor representations. The restrictions of Φ to $D \otimes 1$ and $1 \otimes A'$ are $*$ -homomorphisms, for every $D \subseteq A \cap B$. Consequently

$$\rho_B(d)V = Vd \quad (d \in D), \quad \rho_{A'}(a)V = Va \quad (a \in A').$$

For every $b \in B$, trace preservation and orthogonality of E_A give

$$\|\rho_B(b)V - Vb\|_{\text{HS}}^2 = 2\|b - E_A(b)\|_{\text{HS}}^2.$$

Put $P = VV^*$ and $h = \int_{U(B)} \rho_B(b)P\rho_B(b)^* db$. The operator P commutes with $\rho_{A'}(A')$, so h commutes with both factor representations. Averaging is an orthogonal projection for the Hilbert–Schmidt inner product; hence

$$\text{Tr}(h - h^2) = \|h - P\|_{\text{HS}}^2 = \int_{U(B)} (d - \|E_A(b)\|_{\text{HS}}^2) db \leq \varepsilon^2 d.$$

Let $Q = 1_{[1/2, 1]}(h)$ and $m = \text{rank } Q$. Since $\text{Tr } h = d$ and $\text{Tr}(QP) = \text{Tr}(Qh)$, functional calculus gives

$$|m - d| \leq \|Q - P\|_{\text{HS}}^2 = \text{Tr } \min\{h, 1 - h\} \leq 2\varepsilon^2 d.$$

In particular $\|(1 - Q)V\|_{\text{HS}}^2 \leq 2\varepsilon^2 d$. Restrict the factor representations to $Q\mathcal{K}$, writing them as $\pi_B, \pi_{A'}$, and put

$$\hat{B} = \pi_B(B), \quad \hat{A} = \pi_{A'}(A').$$

These algebras satisfy $\hat{B} \subseteq \hat{A}$.

Let W be the polar part of QV . Polar decomposition preserves intertwining, so W intertwines A' and every common subalgebra D exactly. Singular-value comparison gives

$$\text{rank } W \geq d - 2\varepsilon^2 d, \quad \|W - V\|_{\text{HS}}^2 \leq 4\varepsilon^2 d.$$

Thus the initial and final complements of W have dimension $O(\varepsilon^2 d)$. Restoring n , we have $m_n/d_n \rightarrow_{\mathcal{U}} 1$; with $\pi_{D,n} = \pi_{B,n}|_{D_n}$, the exact identities are

$$(3.1) \quad \begin{aligned} W_n d &= \pi_{D,n}(d)W_n \quad (d \in D_n), \\ W_n a &= \pi_{A',n}(a)W_n \quad (a \in A'_n). \end{aligned}$$

The preceding estimates also give

$$(3.2) \quad \sup_{b \in B_n, \|b\| \leq 1} d_n^{-1/2} \|\pi_{B,n}(b)W_n - W_n b\|_{\text{HS}} \leq (4 + \sqrt{2})\varepsilon_n \rightarrow_{\mathcal{U}} 0.$$

We record the multiplicity estimates needed below. Write

$$B_n = \bigoplus_i (M_{p_{n,i}}(\mathbb{C}) \otimes 1_{q_{n,i}}), \quad A_n = \bigoplus_j (M_{r_{n,j}}(\mathbb{C}) \otimes 1_{s_{n,j}}).$$

Since $\pi_{B,n}$ and $\pi_{A',n}$ are commuting representations of these finite-dimensional algebras, there are integers $k_{n,ji} \geq 0$ such that

$$Q_n \mathcal{K}_n = \bigoplus_{j,i} (\mathbb{C}^{p_{n,i}} \otimes \mathbb{C}^{s_{n,j}})^{\oplus k_{n,ji}}.$$

Put $r'_{n,j} = \sum_i k_{n,ji} p_{n,i}$ and $q'_{n,i} = \sum_j k_{n,ji} s_{n,j}$. The exact B_n -intertwiner

$$T_n = \int_{U(B_n)} \pi_{B,n}(b) Q_n V_n b^* db$$

satisfies $\|T_n - V_n\|_{\text{HS}} \leq 2\sqrt{2}\varepsilon_n \sqrt{d_n}$, and therefore $\text{rank } T_n \geq d_n - 8\varepsilon_n^2 d_n$. Comparing multiplicities with T_n and W_n , respectively, gives

$$\sum_i p_{n,i} \min\{q_{n,i}, q'_{n,i}\} \geq \text{rank } T_n, \quad \sum_j s_{n,j} \min\{r_{n,j}, r'_{n,j}\} \geq \text{rank } W_n.$$

Both original total dimensions equal d_n , and both corrected totals equal $m_n = d_n + O(\varepsilon_n^2 d_n)$. It follows that

$$(3.3) \quad \frac{1}{d_n} \sum_j s_{n,j} |r'_{n,j} - r_{n,j}| + \frac{1}{d_n} \sum_i p_{n,i} |q'_{n,i} - q_{n,i}| \xrightarrow{\mathcal{U}} 0.$$

Finally, use W_n to identify the two ambient spaces after adding negligible complementary spaces and completing it to a unitary U_n . Extend algebras on added corners by scalars. Equation (3.2) and surjectivity of $\pi_{B,n} : B_n \rightarrow \hat{B}_n$, including on unit balls, give both Hausdorff estimates for B_n and $\hat{B}_n$ under this identification. For A_n , let p_n, q_n be the initial and final supports of W_n . Equation (3.1) gives $p_n \in A_n$, $q_n \in \hat{A}_n$, and, by taking commutants on the support corners,

$$W_n(p_n A_n p_n) W_n^* = q_n \hat{A}_n q_n.$$

Compression to these supports changes contractions by $o_2(1)$ uniformly, and contractions in the corner algebras extend by zero. Thus the same identification gives

$$(3.4) \quad d_{H,2}((B_n)_1, (\hat{B}_n)_1) + d_{H,2}((A_n)_1, (\hat{A}_n)_1) \xrightarrow{\mathcal{U}} 0.$$

Here $d_{H,2}$ is the Hausdorff distance of operator-norm unit balls for the normalized 2-norm. This proves equality of the respective ultraproducts. The exact D_n -equivariance of W_n supplies the relative assertion on its asymptotically full supports; no equivariance of the unitary completion is needed. $\square$

4. The no-drift argument

The conditional median relative to D_n turns the order between the corrected scale operators into equality in the ultraproduct. The following finite-dimensional estimate then gives the reverse near inclusion directly. Remark 4.3 explains why the common internal subalgebra is essential.

LEMMA 4.1. *Let $B \subseteq C \subseteq M_d(\mathbb{C})$ be unital subalgebras, with block decompositions*

$$B = \bigoplus_i (M_{p_i}(\mathbb{C}) \otimes 1_{q_i}), \quad C = \bigoplus_j (M_{r_j}(\mathbb{C}) \otimes 1_{s_j}).$$

Let w_i, z_j be their minimal central projections and put

$$\Delta_B = \sum_i \frac{p_i}{q_i} w_i, \quad \Delta_C = \sum_j \frac{r_j}{s_j} z_j, \quad R = \Delta_C \Delta_B^{-1}.$$

Then $R \geq 1$ and, for the trace-preserving conditional expectation $E_B: C \rightarrow B$,

$$(4.1) \quad \sup_{a \in C, \|a\| \leq 1} \|a - E_B(a)\|_2 \leq \sqrt{2} \operatorname{tr}(1 - R^{-1})^{1/2}.$$

PROOF. Let (k_{ji}) be the Bratteli matrix, so that $r_j = \sum_i k_{ji} p_i$ and $q_i = \sum_j k_{ji} s_j$. On the nonzero cell $z_j w_i$, the value of R is $r_j q_i / (s_j p_i) \geq k_{ji}^2 \geq 1$.

We first compute the mean squared distance to B over $U(C)$. Write the j -th block of $u \in U(C)$ using occurrences (i, α) , $1 \leq \alpha \leq k_{ji}$. Then

$$E_B(u)_i = \frac{1}{q_i} \sum_{j, \alpha} s_j u_{j, (i, \alpha), (i, \alpha)}.$$

The blocks u_j are independent Haar unitaries in $U(r_j)$, and their entries satisfy $\int (u_j)_{ab} \overline{(u_j)_{cd}} du_j = \delta_{ac} \delta_{bd} / r_j$. Consequently,

$$\int_{U(C)} \|E_B(u)\|_2^2 du = \frac{1}{d} \sum_{j, i} \frac{k_{ji} p_i^2 s_j^2}{q_i r_j} = \operatorname{tr}(R^{-1}).$$

Since E_B is an orthogonal projection in L^2 , this gives

$$\delta := \int_{U(C)} \|u - E_B(u)\|_2^2 du = \operatorname{tr}(1 - R^{-1}).$$

For the uniform bound, take commutants in $M_d(\mathbb{C})$ and also write E_B for the ambient conditional expectation. Conditional expectations are Haar averages over the appropriate commutant unitary groups. Thus, for every $v \in U(C)$,

$$\begin{aligned} \|v - E_B(v)\|_2^2 &= \frac{1}{2} \int_{U(B')} \|[v, w]\|_2^2 dw \\ &\leq 2 \int_{U(B')} \|w - E_{C'}(w)\|_2^2 dw = 2\delta. \end{aligned}$$

Here $E_{C'}(w)$ commutes with v , and the last equality follows from Fubini: both integrals defining that equality are one half of $\int_{U(B')} \int_{U(C)} \|[u, w]\|_2^2 du dw$. Finally, every contraction in C is a convex combination of unitaries, so convexity proves (4.1). $\square$

THEOREM 4.2. *Let $A = \prod_{n \rightarrow \mathcal{U}} A_n$ and $D = \prod_{n \rightarrow \mathcal{U}} D_n$ be internal finite-dimensional subalgebras of a tracial matrix ultraproduct, with $D_n \subseteq A_n$. Let $u_1, \dots, u_m \in U(\mathcal{M})$ have representatives $u_{\ell, n} \in D'_n$. Assume $u_\ell^* A u_\ell \subseteq A$ for $1 \leq \ell \leq m$ and $D = A \cap \bigcap_{\ell=1}^m \{u_\ell\}'$. Then $u_\ell^* A u_\ell = A$ for $1 \leq \ell \leq m$.*

PROOF. Write $A_n = \bigoplus_{i \in I_n} (M_{r_{n,i}}(\mathbb{C}) \otimes 1_{s_{n,i}})$, with minimal central projections $z_{n,i}$, and put

$$\Delta_n = \sum_i \frac{r_{n,i}}{s_{n,i}} z_{n,i} \in Z(A_n)_+.$$

For commuting positive invertible operators, write $\beta_S(T) = T(T + S)^{-1}$. If e is a minimal central projection of D_n , then

$$E_{D_n}(e \beta_t(\Delta_n)) = c_{n,e}(t) e.$$

Indeed, this expectation belongs to eD_n and commutes with D_n . The scalar $c_{n,e}(t)$ is continuous and strictly decreasing from 1 to 0. Choose $m_{n,e} > 0$ with $c_{n,e}(m_{n,e}) = 1/2$, and set

$$m_n = \sum_e m_{n,e} e \in Z(D_n)_+, \quad x_n = \beta_{m_n}(\Delta_n).$$

Thus $E_{D_n}(x_n) = \frac{1}{2}1$. We call m_n the D_n -conditional median of Δ_n .

Fix ℓ , and put $A_n^+ = A_n$, $A_n^- = u_{\ell,n}^* A_n u_{\ell,n}$, and $y_n = u_{\ell,n}^* x_n u_{\ell,n}$. Lemma 2.2 gives $A_n^- \subset_{2,\varepsilon_n} A_n^+$ with $\varepsilon_n \rightarrow_{\mathcal{U}} 0$. Since $u_{\ell,n} \in D'_n$, both algebras contain D_n . Apply the relative form of Proposition 3.1, obtaining $\hat{A}_n^- \subseteq \hat{A}_n^+$ on a space of dimension $\hat{d}_n$.

Suppress n from the block indices. Label the source blocks using conjugation by $u_{\ell,n}$, so that their original sizes and multiplicities are again r_i, s_i . The Bratteli matrix (k_{ji}) of the corrected inclusion gives

$$\hat{A}_n^- = \bigoplus_{i:s'_i > 0} (M_{r_i}(\mathbb{C}) \otimes 1_{s'_i}), \quad \hat{A}_n^+ = \bigoplus_{j:r'_j > 0} (M_{r'_j}(\mathbb{C}) \otimes 1_{s_j}),$$

where $r'_j = \sum_i k_{ji} r_i$ and $s'_i = \sum_j k_{ji} s_j$. Equation (3.3) gives

$$(4.2) \quad \frac{1}{d_n} \sum_j s_j |r'_j - r_j| + \frac{1}{d_n} \sum_i r_i |s'_i - s_i| \rightarrow_{\mathcal{U}} 0.$$

In particular, $\hat{d}_n/d_n \rightarrow_{\mathcal{U}} 1$. Write $\hat{w}_i, \hat{z}_j$ for the corrected central projections and set

$$\Delta_n^- = \sum_{i:s'_i > 0} \frac{r_i}{s'_i} \hat{w}_i, \quad \Delta_n^+ = \sum_{j:r'_j > 0} \frac{r'_j}{s_j} \hat{z}_j.$$

On every nonzero cell $\hat{z}_j \hat{w}_i$,

$$(4.3) \quad R_n := \Delta_n^+ (\Delta_n^-)^{-1} = \frac{r'_j s'_i}{r_i s_j} \geq k_{ji}^2 \geq 1.$$

Here the scalar equality describes the restriction of R_n to the cell. Thus $\Delta_n^- \leq \Delta_n^+$.

Let $\pi_{D,n}$ be the common representation of D_n furnished by the correction, and put $\hat{m}_n = \pi_{D,n}(m_n)$ and $x_n^\pm = \beta_{\hat{m}_n}(\Delta_n^\pm)$. The three operators $\Delta_n^-, \Delta_n^+, \hat{m}_n$ commute, since the first is central in $\hat{A}_n^-$, the second is central in $\hat{A}_n^+$, and the third belongs to $\hat{A}_n^-$. Consequently,

$$(4.4) \quad 0 < x_n^- \leq x_n^+ < 1.$$

We verify that these corrected bounded scales represent the original ones. Let W_n be the common partial isometry furnished by Proposition 3.1. Define the raw transported scales on the corrected space by

$$\tilde{x}_n^- = \pi_{B,n}(y_n) = \beta_{\hat{m}_n} \left(\sum_{i:s'_i > 0} \frac{r_i}{s'_i} \hat{w}_i \right), \quad \tilde{x}_n^+ = \beta_{\hat{m}_n} \left(\sum_{j:r'_j > 0} \frac{r'_j}{s_j} \hat{z}_j \right).$$

Here $B_n = A_n^-$, and the source formula uses $u_{\ell,n} m_n = m_n u_{\ell,n}$. Since y_n is a contraction of A_n^- , the uniform estimate (3.2) gives

$$\|\tilde{x}_n^- W_n - W_n y_n\|_2 \rightarrow_{\mathcal{U}} 0.$$

On the target side, W_n exactly intertwines D_n and $(A_n^+)'$. In particular it intertwines m_n and Δ_n , whose central projections have the coordinate standard labels j . Functional calculus on its retained supports therefore gives

$$\tilde{x}_n^+ W_n = W_n x_n.$$

The elementary scalar bound

$$(4.5) \quad \left| \frac{a}{a+s} - \frac{b}{b+s} \right|^2 \leq \left| \frac{a}{b} - 1 \right| \quad (a, b, s > 0)$$

is uniform in s : the difference is at most both 1 and $|a/b - 1|$, since it equals $|a/b - 1|bs/((a+s)(b+s))$. Apply this bound on the full corrected blocks, with $a/b = s_i/s'_i$ on source block i and $a/b = r_j/r'_j$ on target block j . Their dimensions are $r_i s'_i$ and $r'_j s_j$, respectively, so

$$\begin{aligned} \|x_n^- - \tilde{x}_n^-\|_{2, \hat{d}_n}^2 &\leq \frac{1}{\hat{d}_n} \sum_i r_i |s'_i - s_i|, \\ \|x_n^+ - \tilde{x}_n^+\|_{2, \hat{d}_n}^2 &\leq \frac{1}{\hat{d}_n} \sum_j s_j |r'_j - r_j|. \end{aligned}$$

Both bounds tend to zero by (4.2). No uniform bound on m_n or m_n^{-1} is needed.

Use the common unitary completion U_n of W_n from the correction, after stabilization to dimension $d_n^\circ = \max\{d_n, \hat{d}_n\}$, and extend the bounded scales by zero on the added summands. Denote the resulting norm by $\|\cdot\|_{2, \circ}$. The complementary corners have dimension $o_{\mathcal{U}}(d_n)$, so the preceding estimates imply

$$(4.6) \quad \|x_n^- U_n - U_n y_n\|_{2, \circ} + \|x_n^+ U_n - U_n x_n\|_{2, \circ} \rightarrow_{\mathcal{U}} 0.$$

Put $x = (x_n)_{n \rightarrow \mathcal{U}}$. Under this common stable identification, (4.6) identifies $(x_n^-)_{n \rightarrow \mathcal{U}}$ with $u_\ell^* x u_\ell$ and $(x_n^+)_{n \rightarrow \mathcal{U}}$ with x . Thus (4.4) passes to $u_\ell^* x u_\ell \leq x$ in $\mathcal{M}$. The two elements have equal trace, so faithfulness of the trace gives $u_\ell^* x u_\ell = x$. This holds for every ℓ , whence $x \in A \cap \bigcap_\ell \{u_\ell\}' = D$. Coordinatewise conditional expectation now yields

$$x = E_D(x) = (E_{D_n}(x_n))_{n \rightarrow \mathcal{U}} = \frac{1}{2} 1.$$

Using (4.6) once more, we obtain $\|x_n^\pm - \frac{1}{2} 1\|_{2, \hat{d}_n} \rightarrow_{\mathcal{U}} 0$ on the corrected space, since the stabilizing corners have vanishing relative dimension.

It remains to apply Lemma 4.1. Functional calculus for the three commuting operators gives

$$R_n^{-1} = \frac{x_n^-(1 - x_n^+)}{x_n^+(1 - x_n^-)}.$$

The joint spectral measures of (x_n^-, x_n^+) concentrate at $(1/2, 1/2)$, where the scalar function on the right is continuous and equals 1. Thus $R_n^{-1} \rightarrow 1$ in measure. Since $0 \leq R_n^{-1} \leq 1$, bounded convergence gives

$$\widehat{\text{tr}}_n(1 - R_n^{-1}) \rightarrow_{\mathcal{U}} 0.$$

Lemma 4.1 therefore places every contraction of $\hat{A}_n^+$ within $o_{\mathcal{U}}(1)$ in normalized 2-norm of the contraction $E_{\hat{A}_n^-}(a) \in \hat{A}_n^-$, uniformly in a . The common identification (3.4) gives the reverse near inclusion $A_n^+ \subset_{2, o_{\mathcal{U}}(1)} A_n^-$. Passing to the ultraproduct

gives $A \subseteq u_\ell^* A u_\ell$. Together with the assumed inclusion, this proves the theorem. $\square$

REMARK 4.3. The following standard example shows why a finite-level anchor is needed. Let

$$A_n = \bigoplus_{i=1}^n M_{2^i}(\mathbb{C}), \quad \tau_n(x_1, \dots, x_n) = \frac{1}{n} \sum_{i=1}^n \text{tr}_{2^i}(x_i),$$

so that the trace is distributed evenly among the n simple summands. This trace is the restriction of the normalized matrix trace under the realization

$$A_n = \bigoplus_{i=1}^n (M_{2^i}(\mathbb{C}) \otimes 1_{2^{n-i}}) \subseteq M_{n2^n}(\mathbb{C}).$$

Define

$$\theta_n(x_1, \dots, x_n) = (0, x_1 \otimes 1_2, x_2 \otimes 1_2, \dots, x_{n-1} \otimes 1_2).$$

Thus θ_n shifts every summand one place to the right, doubles its defining representation, and annihilates the final summand $M_{2^n}(\mathbb{C})$. Each θ_n is an exact nonunital, noninjective $*$ -homomorphism. For every contraction $x = (x_1, \dots, x_n)$,

$$\|1 - \theta_n(1)\|_{2, \tau_n}^2 = \frac{1}{n}, \quad |\tau_n(\theta_n(x)) - \tau_n(x)| \leq \frac{1}{n},$$

and

$$0 \leq \|x\|_{2, \tau_n}^2 - \|\theta_n(x)\|_{2, \tau_n}^2 = \frac{1}{n} \|x_n\|_{2, \text{tr}_{2^n}}^2 \leq \frac{1}{n}.$$

Consequently $(\theta_n)_n$ induces a unital trace-preserving injective $*$ -endomorphism θ of $\prod_{n \rightarrow \mathcal{U}} (A_n, \tau_n)$. This endomorphism is implemented by a unitary in the ambient tracial ultraproduct. Indeed, let $w_n \in U(n2^n)$ cyclically permute the n summand spaces of dimension 2^n , sending the i -th space to the $(i+1)$ -st and the last space to the first. Choose the identifications so that conjugation by w_n agrees with θ_n on the first $n-1$ input summands. For every contraction $x = (x_1, \dots, x_n) \in A_n$,

$$\|w_n x w_n^* - \theta_n(x)\|_2^2 = \frac{1}{n} \|x_n\|_{2, \text{tr}_{2^n}}^2 \leq \frac{1}{n}.$$

Hence $w = (w_n)_{n \rightarrow \mathcal{U}}$ implements θ in the ambient ultraproduct: $\theta(a) = w a w^*$ for every $a \in \prod_{n \rightarrow \mathcal{U}} (A_n, \tau_n)$. It is not surjective. Indeed, set $v = \text{diag}(1, -1) \in M_2(\mathbb{C})$ and

$$y_n = (v, 1_2 \otimes v, 1_4 \otimes v, \dots, 1_{2^{n-1}} \otimes v) \in U(A_n).$$

In the first summand the range of θ_n is zero, while in the j -th summand, for $j \geq 2$, it is $M_{2^{j-1}}(\mathbb{C}) \otimes 1_2$. The trace-preserving conditional expectation of $1_{2^{j-1}} \otimes v$ onto this subalgebra is zero. Therefore $\text{dist}_{2, \tau_n}(y_n, \theta_n(A_n)) = 1$ for every n , and $(y_n)_{n \rightarrow \mathcal{U}}$ does not lie in the range of θ .

The lost input and output boundary summands each have trace $1/n$, so their defects disappear in the ultraproduct, while the proper matrix amplification persists in every summand. Thus asymptotic unitality and trace preservation alone do not prevent drift. It is likewise not enough merely to observe that each finite center has finitely many atoms. The internality of D , together with the identity $E_{D_n}(x_n) = \frac{1}{2}1$, provides the finite-level anchor that excludes this phenomenon and is essential to the argument. In the permutation proof of [17], the corresponding role is played by the ambient Kazhdan component decomposition and its weighted median.

REMARK 4.4. The distinction between genuine finite quotients and general sofic approximations is already visible in $BS(1, 2) = \langle a, t \mid tat^{-1} = a^2 \rangle$. In a height approximation, level i may be taken to consist of 2^{n-i} a -cycles of length 2^i , so every level has the same size. Away from the two end levels, t sends each cycle onto one of the two a^2 -orbits in a cycle at the next level. This is the permutation version of the shift in Remark 4.3. In contrast, in the finite quotients $C_{2^{n-1}} \rtimes C_n$, the generator of C_n acts by multiplication by 2, so $\langle a^2 \rangle = \langle a \rangle$ and there is no drift.

This behavior of finite quotients is not a consequence of the abelianity of $\langle a \rangle$. If $q: G \rightarrow Q$ is a homomorphism to a finite group and $tHt^{-1} \leq H$, then $q(t)q(H)q(t)^{-1} \leq q(H)$. The two groups have the same order, so they are equal. Thus a compressor always normalizes the finite image of H .

5. Proof of the main results

5.1. From the centralizer problem to normalization. We now assume a positive answer to the centralizer problem and prove Theorem 1.2.

PROPOSITION 5.1. *Let $H < G$ be Kazhdan groups and let $\pi: G \rightarrow U(\mathcal{M})$ be a homomorphism. Put $A = \pi(H)' \cap \mathcal{M}$ and $D = \pi(G)' \cap \mathcal{M}$. After stable negligible modifications, there are finite-dimensional algebras $D_n \subseteq A_n$ such that $A = \prod_{n \rightarrow \mathcal{U}} A_n$ and $D = \prod_{n \rightarrow \mathcal{U}} D_n$. If $t_1, \dots, t_m \in G$ and $u_\ell = \pi(t_\ell)$, representatives may be chosen so that $u_{\ell,n} \in D'_n$ for all n and ℓ .*

PROOF. Apply the centralizer problem to $\pi|_H$ and $\pi|_G$. Since both groups are Kazhdan, there are finite-dimensional algebras $D_n, A'_n \subseteq M_{d_n}(\mathbb{C})$ such that $A = \prod_{n \rightarrow \mathcal{U}} A'_n$ and $D = \prod_{n \rightarrow \mathcal{U}} D_n$.

Since $D \subseteq A$, Lemma 2.2 gives $D_n \subset_{2, \varepsilon_n} A'_n$ with $\varepsilon_n \rightarrow_{\mathcal{U}} 0$. Apply Proposition 3.1 to this near inclusion and use the common stable identification supplied by that proposition. Denote the corrected algebras by $D_n \subseteq A_n$; they still represent $D \subseteq A$.

Now choose unitary representatives $v_{\ell,n}$ of u_ℓ in the corrected ambient dimensions. Since $D \subseteq \{u_\ell\}'$, a diagonal argument gives $\sup_{d \in U(D_n)} \|[v_{\ell,n}, d]\|_2 \rightarrow_{\mathcal{U}} 0$. Indeed, otherwise one could choose $d_n \in U(D_n)$ for which the commutators stay bounded away from zero; then $(d_n)_{n \rightarrow \mathcal{U}} \in D$ would not commute with u_ℓ . Haar averaging $v_{\ell,n}$ under conjugation by $U(D_n)$ gives an operator in D'_n at $o_2(1)$ -distance from $v_{\ell,n}$. Its polar part and support projections also lie in D'_n . Complete the polar part blockwise on its initial and final kernels to a unitary $u_{\ell,n} \in D'_n$. Since $v_{\ell,n}$ is unitary, the polar-decomposition estimate gives $\|u_{\ell,n} - v_{\ell,n}\|_2 \rightarrow_{\mathcal{U}} 0$. Thus the new sequence still represents u_ℓ . $\square$

PROOF OF THEOREM 1.2. The group G is finitely generated, and P_H generates G . Consequently there are $t_1, \dots, t_m \in P_H$ such that $G = \langle H, t_1, \dots, t_m \rangle$. Put $A = \pi(H)' \cap \mathcal{M}$ and $D = \pi(G)' \cap \mathcal{M}$. Apply Proposition 5.1 and choose representatives $u_{\ell,n} \in D'_n$ of $u_\ell = \pi(t_\ell)$. The choice of generators gives $D = A \cap \bigcap_{\ell=1}^m \{u_\ell\}'$. Thus Proposition 5.1 supplies all the hypotheses of Theorem 4.2 except the one-sided inclusions.

Let $x \in A$ and $h \in H$. Put $k = t_\ell h t_\ell^{-1} \in H$. Then $[u_\ell^* x u_\ell, \pi(h)] = u_\ell^* [x, \pi(k)] u_\ell = 0$. Thus $u_\ell^* A u_\ell \subseteq A$. By Theorem 4.2, equality holds for every ℓ . The algebra A is fixed pointwise by $\pi(H)$ and normalized by all u_ℓ , hence it is normalized by $\pi(G)$. $\square$

5.2. A conditional nonhyperlinear group. We finish the proof of Theorem 1.3.

PROOF OF THEOREM 1.3. Because H is infranormal but not normal, there is a strict compressor $t \in P_H$, that is, $tHt^{-1} < H$. Indeed, if every element of P_H normalized H , then the group generated by P_H would normalize H .

Put $P = G *_H G$. Write G_1 and G_2 for its two factors, and write g_i for the copy of $g \in G$ in G_i . Suppose that P is hyperlinear and choose an injective homomorphism $\pi: P \rightarrow U(\mathcal{M})$. Put $A = \pi(H)' \cap \mathcal{M}$. The element $c = t_1^{-1}t_2$ centralizes H : for $h \in H$, the element $t_2ht_2^{-1}$ belongs to the common subgroup H and hence $t_1^{-1}(t_2ht_2^{-1})t_1 = h$. Thus $\pi(c) \in A$.

Apply Theorem 1.2 to the first copy G_1 . It follows that A is normalized by $\pi(G_1)$, and therefore $\pi(t_2t_1^{-1}) = \pi(t_1ct_1^{-1}) \in A$. Choose $h \in H \setminus tHt^{-1}$. The commutator $[h, t_2t_1^{-1}]$ is nontrivial: after collecting letters from the same factor, its normal form is $(ht_2)(t_1^{-1}h^{-1}t_1)t_2^{-1}$. A strict compressor cannot belong to H , so the first and third letters lie outside H ; the middle letter lies outside H precisely because $h \notin tHt^{-1}$. The word is therefore reduced. On the other hand, $\pi(t_2t_1^{-1}) \in A$ commutes with $\pi(h)$, so this nontrivial commutator lies in the kernel of π , a contradiction.

Kun and the author proved in [17, Theorem E] that, for every prime power q and $r, d \geq 3$, the group $H = \text{EL}_r(\mathbb{F}_q[x_1, \dots, x_d])$ and the group

$$G = \text{EL}_r(\mathbb{F}_q[x_1^{\pm 1}, \dots, x_d^{\pm 1}]) \rtimes \text{SL}_d(\mathbb{Z})$$

are residually finite Kazhdan groups and that H is infranormal but not normal in G . In fact, one may see a strict compressor explicitly. If $u_{12} = I + E_{12} \in \text{SL}_d(\mathbb{Z})$ and $t = (1, u_{12}) \in G$, then $tHt^{-1} = \text{EL}_r(\mathbb{F}_q[x_1, x_1x_2, x_3, \dots, x_d]) < H$. For example, an elementary matrix with coefficient x_2 lies in H but not in tHt^{-1} . $\square$

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