

The unitary duals of $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_3(\mathbb{R})$

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ABSTRACT. This survey gives explicit descriptions of the full unitary duals of $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_3(\mathbb{R})$, with particular attention to the relation between representation parameters, Fell topology, and central eigenvalues.

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1. Introduction

Fourier analysis on a locally compact abelian group is organized by its unitary characters. For a nonabelian group, irreducible unitary representations play the corresponding role. Their equivalence classes form the *unitary dual* $\widehat{G}$, equipped with the Fell topology, which records approximation of matrix coefficients. Even when the irreducible representations can be listed explicitly, understanding this space requires a second step: one must determine how the families in the list approach one another. A family of irreducibles may have several inequivalent limits, so the resulting topology need not be Hausdorff.

The groups $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_3(\mathbb{R})$ provide two examples in which these phenomena can be described completely. In the first, the principal and complementary series join into a single continuous family. At its other end the complementary series converges simultaneously to the trivial representation and to the two discrete series of extreme K -weight ± 2 . The resulting component is a half-line with three origins. The second example has two principal-series parameter spaces, each a closed half-plane, and two complementary strips attached to their boundaries. At the far edge of each strip there is a special limit and a tempered limit; the tempered limits of the two strips form the same line. Here two special representations can be separated even though neither can be separated from the corresponding tempered representation.

These examples also distinguish two ways of extracting Hausdorff quotients from the dual. The maximal Hausdorff quotient identifies precisely the points that every continuous Hausdorff-valued function must identify. The Casimir operators supply particular continuous functions, but they lose additional information. Already for $\mathrm{PSL}_2(\mathbb{R})$ they do not distinguish the two signs of a higher discrete series. For $\mathrm{PSL}_3(\mathbb{R})$, even the trivial representation and a nontrivial tempered representation have the same infinitesimal character. We determine every fiber of the Casimir maps and compare them with the maximal Hausdorff quotients.

Our subject is the *full* unitary dual. The tempered dual, consisting of irreducible representations weakly contained in the regular representation, is a closed subspace. The complementary families belong to the larger space. This distinction is also the distinction between the full and reduced group C^* -algebras; see [7, §1].

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The representation-theoretic input is classical. For $\mathrm{PSL}_2(\mathbb{R})$ it is the rank-one classification of Bargmann, with modern parameter conventions as in [2, 17, 10]. For $\mathrm{PSL}_3(\mathbb{R})$ we use the real general linear classification of Vahutinskiĭ, Speh, and Vogan, in the formulation of Tadić [18, 15, 19, 16]. The topological input is Milićić's theorem, stated explicitly in Vogan's account [20, Theorem 2]. The models proved here are an expository assembly of these results; the particular presentation and Hausdorff-quotient descriptions are deductions from that input, with the arguments supplied below.

Section 2 recalls the tools for reading topology from characters and central eigenvalues. Section 3 treats the rank-one example, which fixes the basic pattern of endpoint degeneration. Sections 4 and 5 give the classification and topology for $\mathrm{PSL}_3(\mathbb{R})$, including a quotient construction and a partition satisfying the frontier condition. Section 6 describes the effect of Casimir invariants and their fibers, and Section 7 places the references in context.

1.1. Conventions. All representations are complex, and unitary representations are strongly continuous. We identify $\mathrm{PSL}_3(\mathbb{R})$ with $\mathrm{SL}(3, \mathbb{R})$, whose center is trivial. For $n = 2, 3$, write $\det_n: \mathrm{GL}(n, \mathbb{R}) \rightarrow \mathbb{R}^\times$ for the determinant character. All parabolic induction is normalized. More explicitly, if $P = LN$ is a parabolic subgroup and σ is a representation of its Levi subgroup L , inflated trivially across N , our convention is

$$f(pg) = \delta_P(p)^{1/2} \sigma(p) f(g), \quad (\mathrm{Ind}_P^G \sigma)(g_0) f(g) = f(gg_0).$$

Here $\delta_P(p)$ is the absolute determinant of the adjoint action of p on the Lie algebra of N . This normalization preserves unitarity for unitary inducing data: the compact realization has Hilbert norm

$$\|f\|^2 = \int_K \|f(k)\|^2 dk,$$

where K is a maximal compact subgroup and dk is its normalized Haar measure.

For general linear groups, $\pi_1 \times \cdots \times \pi_r$ denotes normalized induction from the block upper triangular parabolic with the indicated Levi blocks. In particular, for characters χ_j of $\mathbb{R}^\times$, the expression $\chi_1 \times \chi_2 \times \chi_3$ denotes one induction from the Borel subgroup of $\mathrm{GL}(3, \mathbb{R})$ with character

$$b \mapsto \chi_1(b_{11}) \chi_2(b_{22}) \chi_3(b_{33}).$$

By induction in stages, it is equally the induction of $(\chi_1 \times \chi_2) \otimes \chi_3$ through the $(2, 1)$ parabolic.

We use Harish–Chandra parameters with the usual ρ -shift: the trivial representation has infinitesimal character ρ . Thus parameters for unitary minimal principal series are purely imaginary. Names such as “special” and “degenerate” vary in the literature; the definitions below, rather than the names, fix the families.

The diagrams are schematic models of the topology, not embeddings of non-Hausdorff spaces into the plane. In particular, duplicated boundary points drawn at different heights are not thereby separated in the Fell topology.

2. Topology, characters, and central eigenvalues

2.1. The Fell topology and the group algebra. The full group C^* -algebra $A = C^*(G)$ provides an operator-algebraic description of the topology. Write $\pi(a)$ for the integrated action of $a \in A$ in a unitary representation. For a net of irreducibles,

$$(1) \quad \pi_i \longrightarrow \pi \iff \|\pi(a)\| \leq \liminf_i \|\pi_i(a)\| \quad \text{for every } a \in A.$$

Indeed, the topology is generated by the open conditions $\|\pi(a)\| > r$, for $a \in A$ and $r \geq 0$.

For the real reductive groups considered here, every irreducible unitary representation maps $C^*(G)$ onto the compact operators on its Hilbert space. This is the CCR property, a theorem of Harish–Chandra; see [7, Theorem 4.7]. In particular, individual points of $\hat{G}$ are closed. The spectrum of $C_r^*(G)$, by contrast, is the tempered dual. We use $C^*(G)$ in (1) because the boundary phenomena under consideration involve non-tempered representations.

2.2. Characters and boundary limits. For these groups the closure of a subset of $\widehat{G}$ is determined by convergent sequences; see the discussion before [20, Theorem 2]. Characters give a particularly effective criterion for identifying all the limits of a family at once.

Let $\mathcal{H}(G)$ be the algebra of compactly supported smooth, left and right K -finite test densities. An admissible representation π has a character Θ_π on this algebra, given by the trace of the corresponding finite-rank operator.

THEOREM 2.1 (Milićić's character-limit criterion). *Suppose $\pi_j \in \widehat{G}$ and that Θ_{π_j} converges pointwise on $\mathcal{H}(G)$ to a nonzero functional Θ . Then*

$$\Theta = \sum_{\sigma \in S} n_\sigma \Theta_\sigma,$$

where S is a nonempty finite subset of $\widehat{G}$ and each n_σ is a positive integer. The sequence π_j converges to every member of S , and these are exactly its limits. Conversely, if $\pi_j \rightarrow \sigma_0$ in the Fell topology, some subsequence has a nonzero character limit of this form, with $\sigma_0 \in S$.

This is [20, Theorem 2]; its operator-algebraic source is [12]. It is important to keep the subsequence in the converse: a sequence can converge in the Fell topology without its characters converging. For example, alternating a complementary-series sequence with one of its endpoint constituents has this behavior.

For a parameter-dependent parabolically induced standard module, the characters depend continuously on the inducing character. This holds whether or not the endpoint standard module is unitary. Accordingly, if the interior is irreducible and unitary, its endpoint composition factors are unitary and form exactly the Fell limit set. This is also discussed in [20, Theorem 3] and [13, (8.5a–d)].

2.3. Central eigenvalues and Hausdorff quotients. Central eigenvalues provide a second tool: their continuity will prevent representation parameters from escaping to infinity along a convergent sequence.

LEMMA 2.2 (Continuity of central eigenvalues). *For every $z \in Z(U(\mathfrak{g}_{\mathbb{C}}))$, the scalar $\chi_\pi(z)$ is a continuous function of $\pi \in \widehat{G}$.*

The result goes back to Bernat–Dixmier [5]; a recent proof, explicitly for the full unitary dual, is [6, Lemma 3.1.5]. In the reductive case the Harish–Chandra isomorphism therefore gives a continuous infinitesimal-character map to $\mathfrak{h}_{\mathbb{C}}^*/W$.

DEFINITION 2.3. A *maximal Hausdorff quotient* of a space X is a continuous surjection $q : X \rightarrow X_H$ onto a Hausdorff space through which every continuous map from X to a Hausdorff space factors uniquely and continuously.

We construct these quotients explicitly. A continuous map into a Hausdorff space identifies any two points that cannot be separated by disjoint neighborhoods. This relation need not be transitive, a feature that will occur for $\mathrm{PSL}_3(\mathbb{R})$. In our examples its equivalence closure has a Hausdorff quotient topology, establishing the required universal property.

3. The dual of $\mathrm{PSL}_2(\mathbb{R})$

3.1. Representations and normalization. We describe the representations of $\mathrm{PSL}_2(\mathbb{R})$ as representations of $\mathrm{SL}(2, \mathbb{R})$ on which $-I$ acts trivially. A K -weight refers to the standard rotation parameter on $\mathrm{SO}(2) \subset \mathrm{SL}(2, \mathbb{R})$; consequently only even weights occur for $\mathrm{PSL}_2(\mathbb{R})$.

For the upper triangular minimal parabolic of $\mathrm{SL}(2, \mathbb{R})$ let

$$I(\nu) = \mathrm{Ind}_B^{\mathrm{SL}(2, \mathbb{R})}(\mathrm{diag}(a, a^{-1}) \mapsto |a|^\nu).$$

This even principal series descends to $\mathrm{PSL}_2(\mathbb{R})$. Its unitary principal representations are $I(ir)$, $r \geq 0$; its complementary representations are $I(s)$, $0 < s < 1$. These representations are irreducible, including $I(0)$, and $I(\nu) \cong I(-\nu)$ at these irreducible parameters.

For $m \geq 1$ let D_{2m}^+ and D_{2m}^- be the lowest- and highest-weight discrete series with extreme weights $2m$ and $-2m$, respectively. The remaining irreducible unitary representation is $\mathbf{1}$. This list is exhaustive [2, 17].

To make the Casimir convention explicit, take the standard basis H, E, F of $\mathfrak{sl}_2$ and set

$$\Omega = \frac{1}{4}H^2 + \frac{1}{2}(EF + FE), \quad C = -4\Omega.$$

Then Ω acts on $I(\nu)$ by $(\nu^2 - 1)/4$. Thus the eigenvalue q of C is

$$(2) \quad \begin{array}{c|c} \text{representation} & q \\ \hline I(ir), \ r \geq 0 & 1 + r^2 \\ I(s), \ 0 < s < 1 & 1 - s^2 \\ D_{2m}^\pm, \ m \geq 1 & 4m(1 - m) \\ \mathbf{1} & 0. \end{array}$$

For a parameter table with this normalization of Ω , see [17]; its indexing of discrete series by infinitesimal character differs by one from our indexing by extreme K -weight.

3.2. A half-line with three origins. For $q > 0$ define

$$J(q) = \begin{cases} I(\sqrt{1-q}), & 0 < q < 1, \\ I(i\sqrt{q-1}), & q \geq 1. \end{cases}$$

The change of series at $q = 1$ produces no topological singularity. At the other endpoint the character identity is

$$(3) \quad \Theta_{I(1)} = \Theta_{\mathbf{1}} + \Theta_{D_2^+} + \Theta_{D_2^-}.$$

In particular, $J(q)$ has precisely these three limits as $q \downarrow 0$; this example is given immediately after [20, Theorem 2].

Let L_3 be the quotient of $[0, \infty) \times \{0, +, -\}$ in which the three points over each $q > 0$ are identified, while the three origins remain distinct. Equivalently, a neighborhood of any origin contains that origin and an interval $0 < q < \delta$ of the common ray.

THEOREM 3.1. *There is a homeomorphism*

$$\widehat{\text{PSL}_2(\mathbb{R})} \cong L_3 \sqcup \{D_{2m}^\pm : m \geq 2\},$$

where the second summand is discrete. The three origins of L_3 are $\mathbf{1}, D_2^+, D_2^-$, and its point $q > 0$ is $J(q)$.

PROOF. The classification gives the asserted bijection. Continuity of induced characters, (3), and Theorem 2.1 give the stated limits for bounded parameters, including the junction at $q = 1$. Conversely, Lemma 2.2 implies convergence of the Casimir eigenvalues along a Fell-convergent sequence. Formula (2) then bounds m and the principal parameter r . Extracting a subsequence in one fixed family with convergent parameters and applying Theorem 2.1 leaves exactly the listed limits. Closures are determined by sequences, so this establishes the topology and excludes accumulation among the higher discrete series. $\square$

The terminology at $q = 0$ requires care. The representations $D_2^\pm$ are square-integrable discrete series. The genuine limits of discrete series for $\text{SL}(2, \mathbb{R})$ have extreme weights ± 1 and odd K -weights; $-I$ acts by -1 , so they do not descend to $\text{PSL}_2(\mathbb{R})$.

3.3. Strata and the Hausdorff quotient. The principal series $q \geq 1$ is closed and not open. The complementary series $0 < q < 1$ is open and not closed. Each higher discrete-series point $D_{2m}^\pm$, $m \geq 2$, is open and closed; $D_2^\pm$ and $\mathbf{1}$ are closed but not open. The entire discrete-series subset is closed and not open. The tempered dual consists of the principal series and all the discrete series; it is closed. In particular, the complementary-series closure is

$$\overline{\{I(s) : 0 < s < 1\}} = \{I(s) : 0 < s < 1\} \cup \{I(0), \mathbf{1}, D_2^+, D_2^-\}.$$

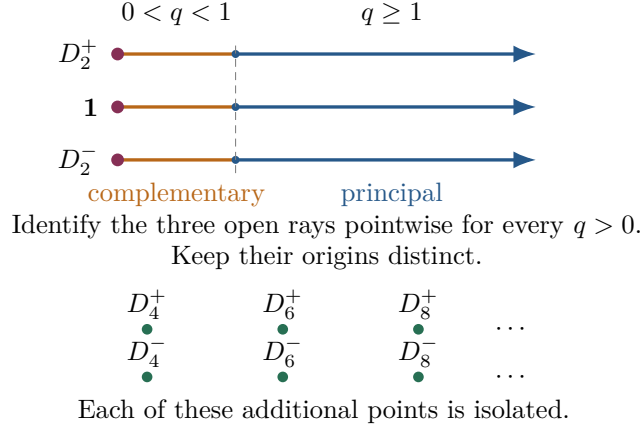


FIGURE 1. The full dual of $\mathrm{PSL}_2(\mathbb{R})$. The three drawn copies of the positive ray represent the same points, not three branches. Each origin is approached by the same complementary sequence.

All three origins must be identified by a map to a Hausdorff space. Identifying them suffices, so

$$(4) \quad (\widehat{\mathrm{PSL}_2(\mathbb{R})})_{\mathrm{H}} \cong [0, \infty) \sqcup \coprod_{m \geq 2} \{D_{2m}^+, D_{2m}^-\}.$$

The Casimir map is not this quotient: it identifies D_{2m}^+ with D_{2m}^- for every $m \geq 2$, although these are distinct isolated points. Its image is

$$[0, \infty) \cup \{4m(1 - m) : m \geq 2\} \subset \mathbb{R}.$$

The fiber at 0 has three points, each negative value in the displayed set has two points, and every positive value has one.

4. The representations of $\mathrm{PSL}_3(\mathbb{R})$

4.1. Reduction to the general linear group. The real cube root gives a group isomorphism

$$(5) \quad \mathrm{GL}(3, \mathbb{R}) \longrightarrow \mathrm{PSL}_3(\mathbb{R}) \times \mathbb{R}^\times, \quad g \longmapsto (\det_3(g)^{-1/3}g, \det_3(g)^{1/3}).$$

Consequently restriction of an irreducible unitary GL_3 representation to $\mathrm{PSL}_3(\mathbb{R})$ is irreducible, every irreducible unitary $\mathrm{PSL}_3(\mathbb{R})$ representation arises in this way, and two restrictions agree precisely when the original representations differ by a unitary determinant character.

The $\mathrm{GL}_n(\mathbb{R})$ classification specializes in degree three to the following five forms:

$$(6) \quad \begin{aligned} &\chi_1 \times \chi_2 \times \chi_3, \quad \delta \times \chi, \quad (\chi \circ \det_2) \times \chi', \\ &(\chi|\cdot|^s) \times (\chi|\cdot|^{-s}) \times \chi', \quad 0 < s < \frac{1}{2}, \quad \chi \circ \det_3. \end{aligned}$$

Here all χ 's are unitary characters of $\mathbb{R}^\times$, and δ is an irreducible unitary GL_2 representation square-integrable modulo its center. Each induction displayed is irreducible and unitary, and these forms exhaust the dual. This follows directly from the unique factorization theorem in [16, pp. 3–4]; see also [19, Theorem 6.18] and [1, Theorem 7.2]. Speh blocks of length greater than one built from GL_2 discrete series, and complementary pairs of blocks of dimension greater than one, cannot occur in degree three. The character blocks of dimensions two and three are already included in (6).

4.2. A nonredundant parameter list. Set

$$\chi_{\epsilon, z}(x) = \mathrm{sgn}(x)^\epsilon |x|^z, \quad E = \{(t_1, t_2, t_3) \in \mathbb{R}^3 : t_1 + t_2 + t_3 = 0\}.$$

For $\epsilon = 0, 1$ put

$$(7) \quad P_\epsilon(t_1, t_2, t_3) = (\chi_{\epsilon, it_1} \times \chi_{\epsilon, it_2} \times \chi_{0, it_3})|_{\mathrm{PSL}_3(\mathbb{R})}.$$

For $\epsilon = 0$ the parameter space is E/S_3 ; for $\epsilon = 1$ it is E/S_2 , where S_2 interchanges the first two coordinates. These are all principal representations. To see the parity convention, a common sign twist is trivial on $\mathrm{PSL}_3(\mathbb{R})$, so every sign triple has a unique representative with even total parity. Such a triple is either 000 or a permutation of 110. The former gives the spherical family, the latter the nonspherical family. All these induced representations are irreducible even when some t_i coincide.

For $t \in \mathbb{R}$, $0 < s < \frac{1}{2}$, and $\epsilon = 0, 1$, put

$$(8) \quad C_\epsilon(t, s) = (\chi_{\epsilon, it+s} \times \chi_{\epsilon, it-s} \times \chi_{0, -2it})|_{\mathrm{PSL}_3(\mathbb{R})}.$$

These are the spherical ($\epsilon = 0$) and nonspherical ($\epsilon = 1$) complementary series, with no further equivalences within the indicated parameter domains.

Let P be the $(2, 1)$ parabolic of $\mathrm{PSL}_3(\mathbb{R})$, whose Levi is identified with $\mathrm{GL}(2, \mathbb{R})$ by

$$A \mapsto \mathrm{diag}(A, \det_2(A)^{-1}).$$

Define the special families by

$$(9) \quad S_\epsilon(t) = \mathrm{Ind}_P^{\mathrm{PSL}_3(\mathbb{R})}(\mathrm{sgn}(\det_2)^\epsilon |\det_2|^{3it}), \quad \epsilon = 0, 1, \quad t \in \mathbb{R}.$$

Both are irreducible and non-tempered; $S_0(t)$ is spherical and $S_1(t)$ is not. Equivalently they are the restrictions of $(\chi_{\epsilon, it} \circ \det_2) \times \chi_{0, -2it}$.

For each integer $k \geq 2$, let D_k be the $\mathrm{GL}_2(\mathbb{R})$ discrete series modulo center, trivial on positive scalar matrices, with

$$D_k|_{\mathrm{SL}_2(\mathbb{R})} = D_k^+ \oplus D_k^-.$$

Here the extreme $\mathrm{SO}(2)$ -weights are k and $-k$; all integers $k \geq 2$ are allowed. Define

$$(10) \quad T_k(t) = \mathrm{Ind}_P^{\mathrm{PSL}_3(\mathbb{R})}(D_k \otimes |\det_2|^{3it}), \quad t \in \mathbb{R}.$$

These are irreducible tempered representations. Since $D_k \otimes \mathrm{sgn}(\det_2) \cong D_k$, there is only one such line per k . The two SL_2 discrete-series signs do not give two $\mathrm{PSL}_3(\mathbb{R})$ families: the disconnected part of the Levi interchanges them. The limiting case of extreme weight 1 is already represented in the nonspherical minimal principal series, not an additional line; see [1, Remark 9.3].

PROPOSITION 4.1. *As a set, the full unitary dual of $\mathrm{PSL}_3(\mathbb{R})$ is the disjoint union*

$$P_0 \sqcup P_1 \sqcup C_0 \sqcup C_1 \sqcup S_0 \sqcup S_1 \sqcup \coprod_{k \geq 2} T_k \sqcup \{\mathbf{1}\},$$

with parameter spaces

$$P_0 = E/S_3, \quad P_1 = E/S_2, \quad C_\epsilon = \mathbb{R} \times (0, \tfrac{1}{2}), \quad S_\epsilon = \mathbb{R}, \quad T_k = \mathbb{R}.$$

PROOF. Restrict the five forms in (6) and remove the common determinant character using (5). Three one-dimensional blocks give (7); a complementary GL_2 block gives (8); a GL_2 character gives (9); a discrete-series block gives (10). A GL_3 character restricts to $\mathbf{1}$. Uniqueness of the general linear building blocks, modulo their order and the common character twist, gives exactly the equivalences already stated. $\square$

The group $\mathrm{PSL}_3(\mathbb{R})$ has no discrete series: its complex rank is 2, whereas the rank of its maximal compact subgroup $\mathrm{SO}(3)$ is 1. The T_k are often called generalized principal series or degenerate tempered representations. The word “degenerate” is also used for (9), so it should not be read as a synonym for tempered.

5. Boundary topology for $\mathrm{PSL}_3(\mathbb{R})$

5.1. The endpoint calculations. The lower endpoint of (8) is the irreducible principal series

$$(11) \quad C_\epsilon(t_j, s_j) \longrightarrow P_\epsilon(t, t, -2t) \quad \text{when } t_j \rightarrow t, \quad s_j \downarrow 0.$$

Write $b_\epsilon(t) = P_\epsilon(t, t, -2t)$. The image of b_0 is the entire boundary of the Weyl chamber E/S_3 : positive and negative t describe its two rays. The image of b_1 is the boundary line of E/S_2 .

Here it is the *equal-parity* pair that has repeated parameters. Repetition on opposite-parity characters gives an interior point of P_1 , except at the zero triple, and creates no additional attachment.

At the upper endpoint the GL_2 standard module $\chi_{\epsilon, it+1/2} \times \chi_{\epsilon, it-1/2}$ has exactly two composition factors,

$$\chi_{\epsilon, it} \circ \det_2, \quad D_2 \otimes |\det_2|^{it}.$$

The second factor is independent of ϵ because its sign twist is equivalent to itself. For the rank-one reducibility and composition formula, see [1, §9.2, (9.1)–(9.5)]. Exactness of induction and irreducibility of the resulting unitary inductions give the character identity

$$(12) \quad \lim_{\substack{t_j \rightarrow t \\ s_j \uparrow 1/2}} \Theta_{C_\epsilon(t_j, s_j)} = \Theta_{S_\epsilon(t)} + \Theta_{T_2(t)}.$$

Thus Theorem 2.1 gives the exact limit set

$$(13) \quad \lim_{\substack{t_j \rightarrow t \\ s_j \uparrow 1/2}} C_\epsilon(t_j, s_j) = \{S_\epsilon(t), T_2(t)\}.$$

5.2. A precise topological model. Define a topology on the set X in Proposition 4.1 as follows. A subset $U \subset X$ is open if and only if its intersection with each listed family is open in that family's ordinary parameter topology and the following collar conditions hold:

- (1) If $b_\epsilon(t) \in U$, then for some $0 < \delta < \frac{1}{2}$ all $C_\epsilon(t', s)$ with $|t' - t| < \delta$ and $0 < s < \delta$ belong to U .
- (2) If $S_\epsilon(t) \in U$, then for some $0 < \delta < \frac{1}{2}$ all $C_\epsilon(t', s)$ with $|t' - t| < \delta$ and $\frac{1}{2} - \delta < s < \frac{1}{2}$ belong to U .
- (3) If $T_2(t) \in U$, then for some $0 < \delta < \frac{1}{2}$ the same upper collar belongs to U for *both* C_0 and C_1 .

These conditions make the T_k -lines for $k \geq 3$ and the point **1** separate open-and-closed components. A neighborhood of a special point need not contain the corresponding tempered point, or conversely. Ordinary rectangular neighborhoods with rational widths, together with the indicated collars, give a countable local basis.

THEOREM 5.1. *The parameter bijection $X \rightarrow \widehat{\mathrm{PSL}_3(\mathbb{R})}$ is a homeomorphism. In particular, (11) and (13) give all attachments between distinct families.*

PROOF. Continuity of induced characters and Theorem 2.1 establish ordinary parameter convergence and the displayed endpoint limits. It remains to exclude any further limits, including escape in the discrete index.

Use unordered Harish–Chandra parameters $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with $\sum \lambda_i = 0$. For the listed families they are

$$(14) \quad \begin{array}{c|c} P_\epsilon(t_1, t_2, t_3) & (it_1, it_2, it_3) \\ C_\epsilon(t, s) & (it + s, it - s, -2it) \\ S_\epsilon(t) & (it + \frac{1}{2}, it - \frac{1}{2}, -2it) \\ T_k(t) & (it + \frac{k-1}{2}, it - \frac{k-1}{2}, -2it) \\ \mathbf{1} & (1, 0, -1). \end{array}$$

These are the usual infinitesimal characters of the inducing blocks, with their common central component removed; compare [16, Lemma 4.2].

For a Fell-convergent sequence, Lemma 2.2 makes $p_2 = \sum \lambda_i^2$ and $p_3 = \sum \lambda_i^3$ bounded. The coordinates λ_i are the roots of

$$(15) \quad z^3 - \frac{1}{2}p_2z - \frac{1}{3}p_3 = 0.$$

A bound on the coefficients of a monic polynomial bounds its roots. Formula (14) therefore bounds all real parameters and the integer k . After extraction, the sequence lies in one fixed family and its parameters converge, allowing $s = 0$ or $s = \frac{1}{2}$ for a complementary family. Its character limit is the corresponding irreducible character except at the upper complementary

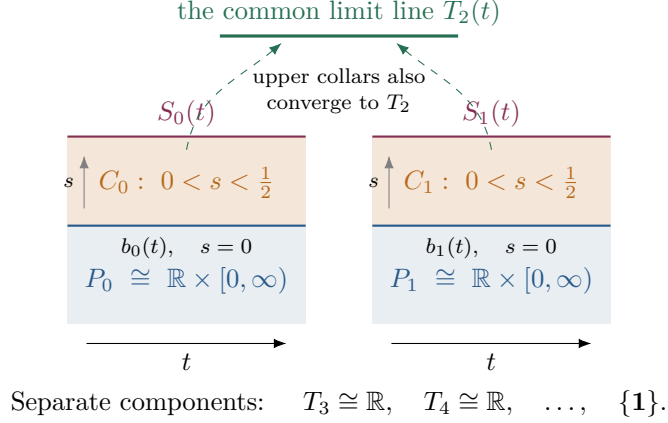


FIGURE 2. The full dual of $\mathrm{PSL}_3(\mathbb{R})$. Each principal piece has been straightened into a half-plane. At the upper edge, a complementary sequence converges both to its own special line and to the common line T_2 . The dashed arrows indicate additional limits, not paths or point identifications.

endpoint, where it is (12). Theorem 2.1 forces the proposed Fell limit to be exactly one of the listed constituents.

For completeness, this subsequence argument identifies the closure operators: any Fell limit of a sequence in a subset has an extracted parameter limit of the specified kind, hence belongs to its closure in X . Conversely, a point in the closure of a subset of X can be approached in one of the finitely many local families, using the stated collar rules, and the corresponding character limit gives Fell convergence. The space X is first countable, and closures in the unitary dual are determined by sequences. The two topologies consequently agree. $\square$

5.3. A quotient construction of the main component. The neighborhood description can also be obtained by gluing ordinary Hausdorff spaces. For $\epsilon = 0, 1$, form

$$B_\epsilon = (P_\epsilon \sqcup ([0, \tfrac{1}{2}] \times \mathbb{R})) / (b_\epsilon(t) \sim (0, t)),$$

with the quotient topology. The strip has coordinates (s, t) ; its open part $(0, \frac{1}{2}) \times \mathbb{R}$ represents $C_\epsilon(t, s)$. Since b_ϵ identifies $\mathbb{R}$ with the boundary of the principal half-plane, B_ϵ is again a closed half-plane. Its remaining boundary is

$$\partial_+ B_\epsilon = \{(\tfrac{1}{2}, t) : t \in \mathbb{R}\}.$$

Take two copies B_ϵ^S and B_ϵ^T of each B_ϵ . Identify corresponding points away from $\partial_+ B_\epsilon$, leaving their upper boundaries distinct. Label the boundary of B_ϵ^S by $S_\epsilon(t)$. Then identify the upper boundary points with the same parameter t in B_0^T and B_1^T , and label the resulting point $T_2(t)$. Give this space Y the quotient topology, with no further identifications.

An open set containing $S_\epsilon(t)$ must contain an upper collar from its own complementary strip. An open set containing $T_2(t)$ has preimages on both tempered boundaries and must contain upper collars from both strips. At the lower attachments one obtains the principal neighborhoods together with their complementary collars. These are exactly the conditions defining X , so

$$\widehat{\mathrm{PSL}_3(\mathbb{R})} \cong Y \sqcup \coprod_{k \geq 3} T_k \sqcup \{1\}.$$

Figure 2 depicts the same construction, drawing the common tempered boundary separately to keep its two attachments visible.

5.4. Nonseparability and closures. For fixed t , the points $S_0(t)$ and $T_2(t)$ have no disjoint neighborhoods, since every pair of neighborhoods meets the same upper collar of C_0 . Similarly $S_1(t)$ and $T_2(t)$ are inseparable. However, $S_0(t)$ and $S_1(t)$ have disjoint neighborhoods: use their own special lines and their own complementary collars. Thus this is not a boundary

with three pairwise inseparable copies. In contrast to the rank-one example, nonseparability here is not transitive.

More explicitly,

$$(16) \quad \overline{C_\epsilon} = C_\epsilon \cup b_\epsilon(\mathbb{R}) \cup S_\epsilon \cup T_2.$$

Each P_ϵ , S_ϵ , and T_k is closed. No sequence with $k \rightarrow \infty$ has a Fell limit. The open/closed status of the named families is therefore as follows.

Subset of $\widehat{\mathrm{PSL}_3(\mathbb{R})}$	Open	Closed
Principal $P_0 \sqcup P_1$	no	yes
Each complementary family C_ϵ	yes	no
Special $S_0 \sqcup S_1$	no	yes
Degenerate tempered $\coprod_{k \geq 2} T_k$	no	yes
Individual T_k , $k \geq 3$	yes	yes
T_2	no	yes
$\{1\}$	yes	yes

Here the whole tempered dual is $P_0 \sqcup P_1 \sqcup \coprod_{k \geq 2} T_k$, a closed subset. The isolation of 1 also follows from Kazhdan's property (T); see [4] for a proof for $\mathrm{PSL}_3(\mathbb{R})$.

5.5. Refining the families: the frontier condition. The familiar names describe natural families of representations, but their closures suggest a finer partition. Put

$$P = P_0 \sqcup P_1, \quad D = \coprod_{k \geq 2} T_k, \quad S = S_0 \sqcup S_1.$$

The six pieces $P, C_0, C_1, D, S, \{1\}$ are all locally closed: the complementary pieces are open, and the other four are closed. Nevertheless, this partition fails the *frontier condition*, which requires that whenever a piece A meets the closure of a distinct piece B , the whole of A belongs to $\overline{B}$. Indeed, (16) gives

$$\overline{C_\epsilon} \cap P = b_\epsilon(\mathbb{R}), \quad \overline{C_\epsilon} \cap D = T_2, \quad \overline{C_\epsilon} \cap S = S_\epsilon.$$

Each intersection is a nonempty proper subset of the corresponding coarse piece.

For an explicit refinement, let

$$\begin{aligned} P_\epsilon^\circ &= P_\epsilon \setminus b_\epsilon(\mathbb{R}), & V_\epsilon &= \{b_\epsilon(0)\}, \\ W_\epsilon^+ &= b_\epsilon((0, \infty)), & W_\epsilon^- &= b_\epsilon((-\infty, 0)). \end{aligned}$$

Take as pieces

$$P_\epsilon^\circ, \quad W_\epsilon^+, \quad W_\epsilon^-, \quad V_\epsilon, \quad C_\epsilon, \quad S_\epsilon \quad (\epsilon = 0, 1),$$

together with each individual line T_k , $k \geq 2$, and the point $\{1\}$. Splitting the boundary of P_1 at $t = 0$ is optional; we do so for uniform notation.

PROPOSITION 5.2. *This refinement is locally finite, consists of locally closed topological manifolds, and satisfies the frontier condition. In every proper closure incidence, the dimension decreases.*

PROOF. The pieces P_ϵ° and C_ϵ are manifolds of dimension two. The boundary rays and the special and tempered lines have dimension one, and the remaining pieces are points. Local closedness follows from the parameter topology and the preceding open and closed subset statements. The only closures that add points are

$$\begin{aligned} \overline{P_\epsilon^\circ} &= P_\epsilon^\circ \cup W_\epsilon^+ \cup V_\epsilon \cup W_\epsilon^-, \\ \overline{W_\epsilon^\pm} &= W_\epsilon^\pm \cup V_\epsilon, \\ \overline{C_\epsilon} &= C_\epsilon \cup W_\epsilon^+ \cup V_\epsilon \cup W_\epsilon^- \cup S_\epsilon \cup T_2. \end{aligned}$$

Each is a union of whole pieces, and every added piece has smaller dimension. There are only finitely many pieces in the main component; each remaining T_k -line and the trivial point is a separate open component. Every point therefore has a neighborhood meeting only finitely many pieces. $\square$

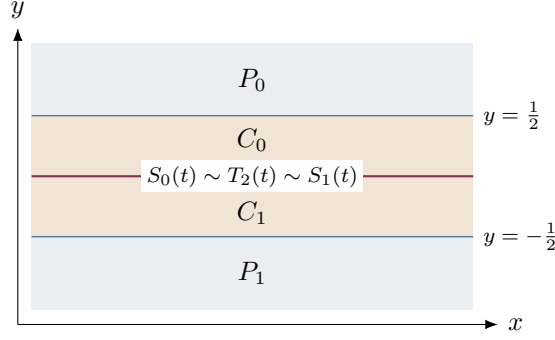


FIGURE 3. The plane component of the maximal Hausdorff quotient of $\widehat{\mathrm{PSL}_3(\mathbb{R})}$. Vertical distances are schematic; the two collars occupy $0 < |y| < \frac{1}{2}$. The quotient also has the separate lines T_k , $k \geq 3$, and the isolated trivial representation.

The frontier condition here concerns a partition of the non-Hausdorff dual. Each individual piece has its ordinary Hausdorff manifold topology.

5.6. The maximal Hausdorff quotient.

THEOREM 5.3. *The maximal Hausdorff quotient identifies, for each $t \in \mathbb{R}$, exactly the three points $S_0(t), T_2(t), S_1(t)$, and makes no other identifications. It is homeomorphic to*

$$(17) \quad (\widehat{\mathrm{PSL}_3(\mathbb{R})})_{\mathrm{H}} \cong \mathbb{R}^2 \sqcup \coprod_{k \geq 3} \mathbb{R} \sqcup \{1\}.$$

PROOF. Every map to a Hausdorff space identifies each inseparable pair, hence all three points with the same t . We show that their quotient is Hausdorff by giving explicit coordinates on its main component.

For P_0 , choose the representative $t_1 \geq t_2 \geq t_3$ and put

$$a = t_1 - t_2 \geq 0, \quad b = t_2 - t_3 \geq 0, \quad x = (b - a)/3, \quad u = \min(a, b).$$

This is a homeomorphism $P_0 \rightarrow \mathbb{R} \times [0, \infty)$. Indeed, for $x \geq 0$ its inverse has $a = u$, $b = u + 3x$, and for $x \leq 0$ it has $a = u - 3x$, $b = u$. On the boundary $b_0(t)$ the coordinate x is precisely t .

For P_1 , use

$$x = (t_1 + t_2)/2, \quad u = |t_1 - t_2|/2;$$

this is again a homeomorphism onto $\mathbb{R} \times [0, \infty)$, with $b_1(t)$ mapped to $(t, 0)$. Extend both coordinate systems over the complementary strips by $(x, u) = (t, -s)$. After identifying the three upper-edge points, put

$$(x, y) = \begin{cases} (x, u + \frac{1}{2}), & \text{on the spherical side,} \\ (x, -u - \frac{1}{2}), & \text{on the nonspherical side.} \end{cases}$$

The common boundary has $y = 0$. The collar neighborhood rules show that this is a homeomorphism of the quotient main component with $\mathbb{R}^2$. The remaining components were already separate Hausdorff lines and a point. Since the quotient is Hausdorff and every Hausdorff-valued map is constant on its fibers, the quotient topology gives the universal property. $\square$

6. Casimir coordinates for $\mathrm{PSL}_3(\mathbb{R})$

6.1. Generators and formulas. The Harish–Chandra isomorphism identifies the center of $U(\mathfrak{sl}_3(\mathbb{C}))$ with the symmetric polynomials in $\lambda_1, \lambda_2, \lambda_3$, subject to $\sum \lambda_i = 0$. Choose generators whose eigenvalues are

$$(18) \quad c_2(\lambda) = \sum_i \lambda_i^2 - 2, \quad c_3(\lambda) = i \sum_i \lambda_i^3.$$

The constant makes the quadratic eigenvalue of $\mathbf{1}$ zero. The factor i in the cubic coordinate is essential for real values on the unitary dual: the unmultiplied real cubic generator is skew-adjoint. Equivalently, the Hermitian condition gives $\lambda \sim -\bar{\lambda}$, so that p_2 is real and p_3 purely imaginary. These choices specify a normalization; other conventional Casimirs differ by the corresponding changes of coordinates.

For principal representations (14) gives

$$(19) \quad (c_2, c_3) = \left(-2 - \sum_i t_i^2, \sum_i t_i^3 \right).$$

All the paired parameters have one common formula. Put

$$(20) \quad F(a, t) = (2a^2 - 6t^2 - 2, -6t(a^2 + t^2)).$$

Substitution of $(a + it, -a + it, -2it)$ into (18) yields

$$(21) \quad \begin{aligned} C_\epsilon(t, s) &\mapsto F(s, t), & S_\epsilon(t) &\mapsto F(\tfrac{1}{2}, t), \\ T_k(t) &\mapsto F(\tfrac{k-1}{2}, t), & \mathbf{1} &\mapsto (0, 0). \end{aligned}$$

Together with (19) and Theorem 5.1, these formulas describe the unitary dual as a topological space over $\mathbb{R}^2$.

6.2. The image and every fiber. For the principal series, set $A = \sum t_i^2$ and $B = \sum t_i^3$. The three real numbers t_i are the roots of $z^3 - (A/2)z - B/3$. Its discriminant is $A^3/2 - 3B^2$. Thus the principal image is exactly the closed cusp

$$(22) \quad \mathcal{P} = \{(x, y) : x \leq -2, \quad 6y^2 \leq [-(x+2)]^3\}.$$

The vertex $(-2, 0)$ corresponds to Harish-Chandra parameter 0.

For $a > 0$, the map F is injective. Indeed its invariants recover the unordered triple $(a + it, -a + it, -2it)$, in which the unique purely imaginary member is $-2it$ and the positive real part is a . Its image is disjoint from $\mathcal{P}$, since the other two members have nonzero real parts. Therefore the complete Casimir image is

$$(23) \quad \mathcal{P} \cup F((0, \tfrac{1}{2}] \times \mathbb{R}) \cup \bigcup_{k \geq 3} F(\{\tfrac{k-1}{2}\} \times \mathbb{R}).$$

The trivial representation adds no new image point: $F(1, 0) = (0, 0)$ is already the image of $T_3(0)$.

For a principal infinitesimal character, the parity multiplicity can be counted without any analytic calculation. The minimal compact Levi has four characters, represented by 000, 110, 101, 011. For a fixed unordered triple (t_i) , take their orbits under the stabilizer of that triple in S_3 . Three distinct coordinates give four orbits; exactly two equal coordinates give three; three equal coordinates give two. Combining this with injectivity of F gives the full fiber table:

Casimir value	Fiber cardinality
Interior of $\mathcal{P}$	4
Boundary of $\mathcal{P}$, excluding its vertex	3
Vertex $(-2, 0)$	2
$F(a, t)$ with $0 < a < \frac{1}{2}$	2
$F(\frac{1}{2}, t)$	3
$F(\frac{k-1}{2}, t)$, $k \geq 3$, except $(0, 0)$	1
$(0, 0)$	2
Outside (23)	0

At $F(\frac{1}{2}, t)$ the fiber is precisely $\{S_0(t), S_1(t), T_2(t)\}$, and at $(0, 0)$ it is $\{\mathbf{1}, T_3(0)\}$.

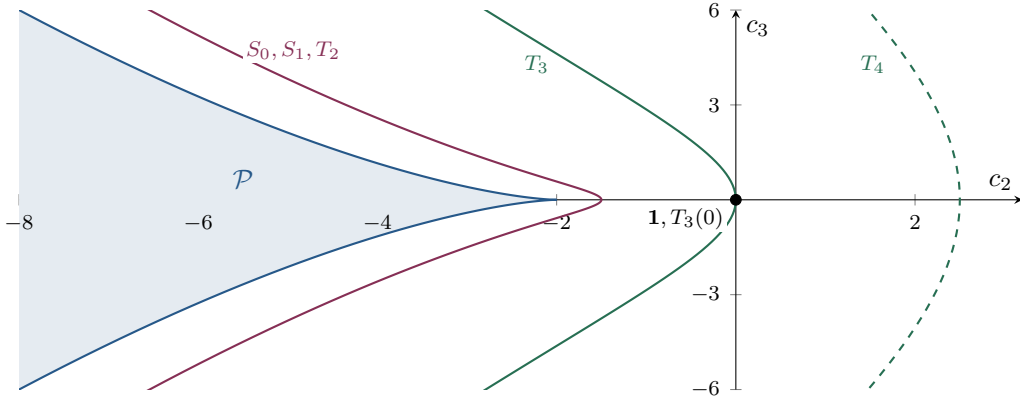


FIGURE 4. Part of the joint Casimir image for $\mathrm{PSL}_3(\mathbb{R})$. The shaded cusp is the principal image. The region between its boundary and the solid magenta curve is the image of both complementary families. That magenta curve is the common image of S_0, S_1, T_2 ; the green curves shown are T_3, T_4 . Further T_k curves continue the pattern. This is a picture of the image, not of the dual: the fiber multiplicities are listed in the text.

6.3. Comparison with Hausdorffization. By continuity, the Casimir map factors through $(\widehat{\mathrm{PSL}_3(\mathbb{R})})_{\mathrm{H}}$. It is not injective there. Most visibly, $\mathbf{1}$ and $T_3(0)$ have the same infinitesimal character $(1, 0, -1)$ and the same Casimir values $(0, 0)$, but are in different open-and-closed components. Their characteristic-component functions separate them continuously. Similarly, principal parity distinctions and the two complementary families survive Hausdorffization, although they have the same central eigenvalues. The only Casimir fibers that are collapsed in forming the maximal Hausdorff quotient are the triples $\{S_0(t), S_1(t), T_2(t)\}$.

7. Bibliographic context

Vahutinskii’s 1968 paper [18] gives a direct classification for $\mathrm{GL}(3, \mathbb{R})$, with the final list in §6. Its proof uses Kirillov’s assertion that an irreducible unitary general linear representation stays irreducible on the mirabolic subgroup. The original proof of that assertion was incomplete. Tadić explains this history in the introduction to [16]; Baruch’s work [3] supplies the completed proof. Vogan’s classification [19] gives an independent approach.

Šijački’s paper [14] treats the universal covering group of $\mathrm{PSL}_3(\mathbb{R})$ and supplies explicit matrix-element constructions. Its genuine covering-group representations must not be added to the dual of $\mathrm{PSL}_3(\mathbb{R})$. Speh’s paper [15] is the directly relevant low-dimensional classification, while [19, 16] place it in the uniform general linear theory. For a modern concise orientation, Ditlevsen’s thesis [8, Paper C, §3.3] lists the five GL_3 families.

Lipsman’s paper [11] is an early reference specifically devoted to dual topology for principal and discrete series. For the complete boundary arguments here, the relevant precise statements are the Milićić criterion and its presentation in [20, Theorems 2–3]. Vogan notes that only part of Milićić’s early work on real semisimple duals was published; the explicit theorem in that account is sufficient for our use.

For an operator-algebraic account of parabolic induction, the paper of Clare–Crisp–Higson [7] provides a complementary perspective. However, its main structure results concern the reduced group C^* -algebra and the tempered dual. The compact-operator theorem recalled there as Theorem 4.7 applies to every irreducible unitary representation and is the full-dual statement used in Section 2. Together these references connect the classical parameter lists with both the character and operator-algebraic descriptions of their topology.

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References

1. A. I. Badulescu and D. Renard, *Unitary dual of $\mathrm{GL}(n)$ at archimedean places and global Jacquet–Langlands correspondence*, Compos. Math. **146** (2010), no. 5, 1115–1164, [arXiv:0905.4143](#); [doi:10.1112/S0010437X10004707](#).
2. V. Bargmann, *Irreducible unitary representations of the Lorentz group*, Ann. of Math. (2) **48** (1947), no. 3, 568–640, [doi:10.2307/1969129](#).
3. E. M. Baruch, *A proof of Kirillov’s conjecture*, Ann. of Math. (2) **158** (2003), no. 1, 207–252, [Article and PDF](#).
4. M. E. B. Bekka and M. Mayer, *On Kazhdan’s property (T) and Kazhdan constants associated to a Laplacian for $\mathrm{SL}(3, R)$* , J. Lie Theory **10** (2000), no. 1, 93–105, [doi:10.5802/jolt.190](#).
5. P. Bernat and J. Dixmier, *Sur le dual d’un groupe de Lie*, C. R. Acad. Sci. Paris **250** (1960), 1778–1779.
6. J. Bradd, N. Higson, and R. Yuncken, *Operator K -theory and tempiric representations*, 2024, Preprint, [arXiv:2412.18924](#); references are to version 1.
7. P. Clare, T. Crisp, and N. Higson, *Parabolic induction and restriction via C^* -algebras and Hilbert C^* -modules*, Compos. Math. **152** (2016), no. 6, 1286–1318, [arXiv:1409.8654v3](#); theorem numbers follow this version. [Published version](#).
8. J. Ditlevsen, *Restricting representations of $\mathrm{GL}(3, \mathbb{R})$ to $\mathrm{GL}(2, \mathbb{R})$* , Ph.D. thesis, Aarhus University, August 2023, [Dissertation PDF](#).
9. A. Henriques, *What is the unitary dual of $\mathrm{PSL}(3, \mathbb{R})$?*, MathOverflow, question 515373, 2026, [Question](#); accessed September 23, 2026.
10. A. W. Knap, *Representation theory of semisimple groups: An overview based on examples*, Princeton Mathematical Series, vol. 36, Princeton University Press, Princeton, NJ, 1986, [Author’s book page](#).
11. R. L. Lipsman, *The dual topology for the principal and discrete series on semisimple groups*, Trans. Amer. Math. Soc. **152** (1970), 399–417, MR0269778.
12. D. Milićić, *On C^* -algebras with bounded trace*, Glasnik Mat. Ser. III **8(28)** (1973), 7–22.
13. S. A. Salamanca-Riba and D. A. Vogan, Jr., *On the classification of unitary representations of reductive Lie groups*, Ann. of Math. (2) **148** (1998), no. 3, 1067–1133, [arXiv:math/9811189](#).
14. Dj. Šijački, *The unitary irreducible representations of $\overline{\mathrm{SL}}(3, R)$* , J. Math. Phys. **16** (1975), no. 2, 298–311, [doi:10.1063/1.522541](#).
15. B. Speh, *The unitary dual of $\mathrm{GL}(3, \mathbb{R})$ and $\mathrm{GL}(4, \mathbb{R})$* , Math. Ann. **258** (1981), no. 2, 113–133, [doi:10.1007/BF01450529](#).
16. M. Tadić, *$\mathrm{GL}(n, \mathbb{C})^\wedge$ and $\mathrm{GL}(n, \mathbb{R})^\wedge$* , Automorphic Forms and L -Functions II. Local Aspects, Contemporary Mathematics, vol. 489, American Mathematical Society, Providence, RI, 2009, [Author’s preprint](#); page references in the text refer to this version, pp. 285–313.
17. P. Trapa, *Irreducible Harish-Chandra modules for $\mathrm{SL}(2, \mathbb{R})$: reference guide*, 2006, Reference material for the University of Utah minicourse. [PDF](#).
18. I. Ya. Vakhutinskii, *Irreducible unitary representations of the group $\mathrm{GL}(3, R)$ of real matrices of the third order*, Math. USSR-Sb. **4** (1968), no. 2, 273–291, English translation of Mat. Sb. (N.S.) 75(117) (1968), 303–320. [Article and PDF](#).
19. D. A. Vogan, Jr., *The unitary dual of $\mathrm{GL}(n)$ over an archimedean field*, Invent. Math. **83** (1986), no. 3, 449–505, [PDF](#); [doi:10.1007/BF01394418](#).
20. ———, *Isolated unitary representations*, Automorphic Forms and Applications (P. Sarnak and F. Shahidi, eds.), IAS/Park City Mathematics Series, vol. 12, American Mathematical Society, Providence, RI, 2007, [Author’s preprint](#); theorem numbers in the text refer to this version, pp. 379–398.

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